problem 1:

$$x^\mu = L \begin{pmatrix} \sin\theta \sin\varphi \\ \sin\theta \cos\varphi \\ \cos\theta \end{pmatrix}$$

$$dx_1 = L \left(\cos\theta \sin\varphi \, d\theta + \sin\theta \cos\varphi \, d\varphi \right)$$

$$dx_2 = L \left(\cos\theta \cos\varphi \, d\varphi - \sin\theta \sin\varphi \, d\varphi \right)$$

$$dx_3 = - L \sin\theta \, d\varphi$$

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2$$

$$= L^2 \left(d\theta^2 + \sin^2\theta \, d\varphi \right)$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & \\ & \sin^2\theta \end{pmatrix} L^2$$

problem 2:

$$\frac{\partial x^\mu}{\partial y^1} = \begin{pmatrix} 1 \\ 0 \\ \partial f/\partial x \end{pmatrix} \qquad \frac{d x^\mu}{\partial y^2} = \begin{pmatrix} 0 \\ 1 \\ \partial f/\partial y \end{pmatrix}$$

$$\gamma_{ab} = \begin{pmatrix} 1+\left(\frac{\partial f}{\partial x}\right)^2 & \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \\ 1+\frac{\partial f \partial f}{\partial x \partial y} & 1+\left(\frac{\partial f}{\partial y}\right)^2 \end{pmatrix}$$

$$n_\mu = \epsilon_{\mu\nu\lambda} \frac{\partial x^\nu}{\partial y^a} \frac{\partial x^\lambda}{\partial y^b} \epsilon^{ab} \frac{1}{2\sqrt{\det \gamma}}$$

$$= \epsilon_{\mu\nu\lambda} \frac{\partial x^\nu}{\partial y^1} \frac{\partial x^\lambda}{\partial y^2} \cdot \frac{1}{\sqrt{\det \gamma}}$$

$$= \left(\frac{\partial x^\nu}{\partial y^1}\right) \times \left(\frac{\partial x^\lambda}{\partial y^2}\right) \cdot \frac{1}{\sqrt{\det \gamma}}$$

$$= \begin{pmatrix} -\partial f/\partial x \\ -\partial f/\partial y \\ 1 \end{pmatrix} \frac{1}{\sqrt{\det \gamma}}$$

$$\det \gamma = 1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$$

$$\hookrightarrow \quad n_\mu n^\mu = 1$$

$$M_{ab} = n_\mu \frac{\partial^2 x^\mu}{\partial y^a \partial y^b}$$

$$= \frac{\partial^2 f}{\partial y^a \partial y^b} \frac{1}{\sqrt{\det \gamma}}$$

$$-\frac{R}{2} = \frac{\det M}{\det \gamma} = \frac{f_{xx} f_{yy} - f_{xy}^2}{\left(1 + (\partial f / \partial x)^2 + (\partial f / \partial y)^2\right)^2}$$

$$= k$$

problem 3:

$$-\frac{R}{2} = \frac{\det M}{\det \gamma} = K$$

invariant under coord. trafos
in 2-space

M_{ab}

γ_{ab}

$y \to y'$

$$\gamma_{ab}' = \frac{\partial y^c \partial y^d}{\partial y'^a \partial y'^b} \gamma_{cd}$$

$$\det \gamma_{ab}' = \det \gamma \cdot \left(\det \frac{\partial y}{\partial y'}\right)^2$$

$$\det M_{ab}' \overset{!}{=} \det M \left(\det \frac{\partial y}{\partial y'}\right)^2$$

This would be true if M_{ab} was a tensor!

$$M_{ab} = \frac{\partial^2 x^\mu}{\partial y^a \partial y^b} n_\mu$$

$$\frac{\partial x^\mu}{\partial y^a} \xrightarrow{y \to y'} \frac{dy'^b}{dy^a} \frac{\partial x^\mu}{\partial y'^b}$$

It's a covariant vector w.r.t 2-space

$$\frac{\partial x^\mu}{\partial y^a} \xrightarrow{x \to x'} \frac{\partial x^\mu}{\partial x'^\nu} \frac{\partial x'^\nu}{\partial y^a}$$

It's a contravariant vector w.r.t. 3-space

$$\frac{\partial x^\mu}{\partial y^a} = T^\mu{}_a$$

In order to get a tensor, we would have
to use the covariant derivative w.r.t 2-space:

$$D_b \frac{\partial x^\mu}{\partial y^a}$$

$$D_b \frac{\partial x^\mu}{\partial y^b} = \frac{\partial^2 x^\mu}{\partial y^a \partial y^b} - \Gamma^c_{ab} \frac{\partial x^\mu}{\partial y^c}$$

$$\Gamma^c_{ab} = \gamma^{cd} \frac{\partial x^\alpha}{\partial y^d} \frac{\partial x^\beta}{\partial y^a \partial y^b} \eta_{\alpha\beta}$$

$$\hookrightarrow D_b \frac{\partial x^\mu}{\partial y^a} = \left(\delta^\mu{}_\nu - P^\mu{}_\nu\right) \frac{\partial x^\nu}{\partial y^a \partial y^b}$$

$$\hookrightarrow D_b \frac{\partial x^\mu}{\partial y^a} n_\mu = \frac{\partial x^\mu}{\partial y^a \partial y^b} n_\mu = M_{ab}$$

So M_{ab} is indeed a tensor!