Problem 1:

$$[\nabla_\alpha, \nabla_\beta] V^\lambda$$

$$= \nabla_\alpha \left(\partial_\beta V^\lambda + \Gamma^\lambda{}_{\beta\kappa} V^\kappa \right) - (\alpha \leftrightarrow \beta)$$

$$= \partial_\alpha \partial_\beta V^\lambda - \Gamma^\kappa{}_{\beta\alpha} \partial_\kappa V^\lambda + \Gamma^\lambda{}_{\alpha\rho} \partial_\beta V^\rho$$

$$+ (\partial_\alpha \Gamma^\lambda{}_{\beta\kappa}) V^\kappa + \Gamma^\lambda{}_{\beta\kappa} \partial_\alpha V^\kappa$$

= symmetric -> drops out

$$+ \Gamma^\lambda{}_{\alpha\rho} \Gamma^\rho{}_{\beta\kappa} V^\kappa - \Gamma^\rho{}_{\beta\alpha} \Gamma^\lambda{}_{\rho\kappa} V^\kappa$$

$$- (\alpha \leftrightarrow \beta)$$

$$= \left(\partial_\alpha \Gamma^\lambda{}_{\beta\kappa} - \partial_\beta \Gamma^\lambda{}_{\alpha\kappa} + \Gamma^\lambda{}_{\alpha\rho} \Gamma^\rho{}_{\beta\kappa} - \Gamma^\lambda{}_{\beta\rho} \Gamma^\rho{}_{\alpha\kappa} \right) V^\kappa$$

$$= - R^\lambda{}_{\kappa\alpha\beta} V^\kappa$$

<u>Likewise:</u> $[\nabla_\alpha, \nabla_\beta] T_{\mu\nu}$

$$= R^\rho{}_{\mu\alpha\beta} T_{\rho\nu} + R^\rho{}_{\nu\alpha\beta} T_{\mu\rho}$$

$\hookrightarrow$ $[\nabla_\alpha, \nabla_\beta] g_{\mu\nu} = 0$ $\quad (\nabla_\alpha g_{\mu\nu} = 0)$

$$= R_{\nu\mu\alpha\beta} + R_{\mu\nu\alpha\beta}$$

$\Rightarrow$ $R_{\mu\nu\alpha\beta}$ antisymmetric in $\mu \leftrightarrow \nu$

Problem 2:

$$g_{\mu\nu} = \frac{\partial x^\alpha \partial x^\beta}{\partial y^\mu \partial y^\nu} \eta_{\alpha\beta}$$

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\kappa} \left\{ \frac{\partial g_{\kappa\lambda}}{\partial y^\nu} + \frac{\partial g_{\nu\kappa}}{\partial y^\lambda} - \frac{\partial g_{\nu\lambda}}{\partial y^\kappa} \right\}$$

$$\frac{\partial g_{\kappa\lambda}}{\partial y^\nu} = \left(\frac{\partial^2 x^\alpha}{\partial y^\kappa \partial y^\nu} \frac{\partial x^\beta}{\partial y^\lambda} + \frac{\partial x^\alpha}{\partial y^\kappa} \frac{\partial^2 x^\beta}{\partial y^\lambda \partial y^\nu} \right) \eta_{\alpha\beta}$$

$$\begin{array}{llll} + & \cancel{\kappa\nu - \lambda} & + & \kappa - \lambda\nu \\ + & \nu\lambda - \kappa & + & \cancel{\nu - \lambda\kappa} \\ - & \cancel{\nu\kappa - \lambda} & - & \cancel{\nu - \lambda\kappa} \end{array}$$

$$\Gamma^\mu_{\nu\lambda} = g^{\mu\kappa} \frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\lambda} \frac{\partial x^\beta}{\partial y^\kappa} \eta_{\alpha\beta}$$

$$R_{\mu\nu\kappa\lambda} \ni \frac{1}{2} \left\{ \frac{\partial^2 g_{\mu\kappa}}{\partial y^\nu \partial y^\lambda} + 3 \text{ perm} \right\}$$

$$\frac{\partial^2 g_{\mu\kappa}}{\partial y^\nu \partial y^\lambda} = \frac{\partial^3 x^\alpha}{\partial y^\mu \partial y^\nu \partial y^\lambda} \frac{\partial x^\beta}{\partial y^\kappa} \eta_{\alpha\beta} \; * $$

$$+ \frac{\partial^3 x^\beta}{\partial y^\kappa \partial y^\nu \partial y^\lambda} \frac{\partial x^\alpha}{\partial y^\mu} \eta_{\alpha\beta} \; *$$

$$+ \frac{\partial^2 x^\alpha}{\partial y^\mu \partial y^\nu} \frac{\partial^2 x^\beta}{\partial y^\kappa \partial y^\lambda} \eta_{\alpha\beta} \qquad *$$

$$+ \frac{\partial^2 x^\alpha}{\partial y^\mu \partial y^\lambda} \frac{\partial^2 x^\beta}{\partial y^\kappa \partial y^\nu} \eta_{\alpha\beta}$$

$$R_{\mu\nu\kappa\lambda} = \cancel{k} \left\{ \frac{\partial^2 x^\alpha}{\partial y^\mu \partial y^\lambda} \frac{\partial^2 x^\beta}{\partial y^\nu \partial y^\kappa} - \frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\lambda} \frac{\partial^2 x^\beta}{\partial y^\kappa \partial y^\mu} \right.$$

$$\left. - \cancel{\frac{\partial^2 x^\alpha}{\partial y^\mu \partial y^\kappa} \frac{\partial^2 x^\beta}{\partial y^\nu \partial y^\lambda}} + \cancel{\frac{\partial x^\alpha}{\partial y^\nu \partial y^\kappa} \frac{\partial x^\beta}{\partial y^\mu \partial y^\lambda}} \right\} \eta_{\alpha\beta}$$

$$R_{\mu\nu\kappa\lambda} = g_{\rho\sigma} \left(\Gamma^\rho_{\mu\kappa} \Gamma^\sigma_{\nu\lambda} - \Gamma^\rho_{\mu\lambda} \Gamma^\sigma_{\nu\kappa} \right)$$

$$\Gamma^\rho_{\mu\kappa} = g^{\rho\delta} \frac{\partial^2 x^\alpha}{\partial y^\mu \partial y^\kappa} \frac{\partial x^\beta}{\partial y^\delta} \eta_{\alpha\beta}$$

$$R_{\mu\nu\kappa\lambda} = g^{\rho\sigma} \left\{ \frac{\partial^2 x^\alpha}{\partial y^\mu \partial y^\kappa} \frac{\partial x^\beta}{\partial y^\rho} \frac{\partial^2 x^\delta}{\partial y^\nu \partial y^\lambda} \frac{\partial x^\epsilon}{\partial y^\sigma} \right.$$

v

$$\times \eta_{\alpha\beta} \eta_{\delta\epsilon} \qquad \left. - \frac{\partial^2 x^\alpha}{\partial y^\mu \partial y^\lambda} \frac{\partial x^\beta}{\partial y^\rho} \frac{\partial^2 x^\delta}{\partial y^\nu \partial y^\kappa} \frac{\partial x^\epsilon}{\partial y^\sigma} \right\}$$

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x

$$g^{\rho\sigma} \frac{\partial x^\beta}{\partial y^\rho} \frac{\partial x^\epsilon}{\partial y^\sigma} = \delta^\beta_\alpha \, \delta^\epsilon_\delta \, \eta^{\alpha\delta} = \eta^{\beta\epsilon}$$

$$g^{\rho\sigma} = \frac{\partial y^\rho}{\partial x^\alpha} \frac{\partial y^\sigma}{\partial x^\delta} \eta^{\alpha\delta}$$

$$\eta_{\alpha\beta}\,\eta_{\gamma\epsilon}\,\eta^{\beta\epsilon} = \eta_{\alpha\gamma}$$

$$\eta_{\alpha\gamma}\left\{ \frac{\partial x^\alpha}{\partial y^\mu \partial y^\kappa}\;\frac{\partial x^\gamma}{\partial y^\nu \partial y^\lambda} - \frac{\partial^2 x^\alpha}{\partial y^\mu \partial y^\lambda}\;\frac{\partial^2 x^\gamma}{\partial y^\kappa \partial y^\nu}\right\}$$

$$R_{\mu\nu\kappa\lambda} = \underline{\underline{0}}$$

Problem 3:

Where things go wrong is in the following step

$$\gamma_{\alpha\beta} = \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}\,\eta_{\mu\nu}$$

these are 2×3 matrices!

$$\gamma^{\alpha\beta} = \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu}\,\eta^{\mu\nu}$$

this does not exist for embedding!

$$P^\mu{}_\kappa = \gamma^{\alpha\beta}\,\frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}\,\eta_{\nu\kappa} \neq \delta^\mu{}_\kappa$$

$P^{\mu}{}_{\kappa}$ is a projection operator!

$$P^2 = P \qquad P^{\mu}{}_{\kappa} P^{\kappa}{}_{\lambda} = P^{\mu}{}_{\lambda}$$

$$\gamma^{\alpha\beta} \frac{\partial x^{\mu}}{\partial y^{\alpha}} \underbrace{\frac{\partial x^{\nu}}{\partial y^{\beta}} \eta_{\nu\kappa} \gamma^{\gamma\delta} \frac{\partial x^{\kappa}}{\partial y^{\gamma}}}_{\gamma_{\beta\gamma}} \frac{\partial x^{\rho}}{\partial y^{\delta}} \eta_{\rho\lambda}$$

$$= \underbrace{\gamma^{\alpha\beta} \gamma_{\beta\gamma} \gamma^{\gamma\delta}}_{\gamma^{\alpha\delta}} \frac{\partial x^{\mu}}{\partial y^{\alpha}} \frac{\partial x^{\rho}}{\partial y^{\delta}} \eta_{\rho\lambda} = P^{\mu}{}_{\lambda}$$

$$\underset{3\times 3}{\delta^{\lambda}{}_{\kappa}} \delta^{\kappa}{}_{\mu} = \delta^{\lambda}{}_{\mu}$$

$$\mathrm{Tr}\, P : \quad \delta^{\lambda}_{\mu} P^{\mu}{}_{\lambda} =$$

$$\underbrace{\gamma^{\alpha\beta} \frac{\partial x^{\mu}}{\partial y^{\alpha}} \frac{\partial x^{\kappa}}{\partial y^{\beta}} \underbrace{\eta_{\kappa\rho} \delta^{\rho}{}_{\mu}}_{\eta_{\kappa\mu}}}_{\gamma^{\alpha\beta}}$$

$$= \gamma^{\alpha\beta} \gamma_{\alpha\beta} = \underset{2\times 2}{\delta^{\alpha}{}_{\alpha}} = \text{dim of emb.}$$

$\alpha, \beta, \gamma, \delta \in \{1, 2\}$ $\mu, \nu, \kappa, \lambda \in \{1, 2, 3\}$

$$R_{\alpha\beta\gamma\delta} = \left(\frac{\partial^2 x^\mu}{\partial y^\alpha \partial y^\gamma} \frac{\partial^2 x^\nu}{\partial y^\beta \partial y^\delta} - \frac{\partial^2 x^\mu}{\partial y^\alpha \partial y^\delta} \frac{\partial^2 x^\nu}{\partial y^\beta \partial y^\gamma} \right)$$

$$\times \left(- \eta_{\mu\nu} + P_{\mu\nu} \right)$$

P is a projection on the tangent space:

$$P^\mu{}_\nu \frac{\partial x^\nu}{\partial y^\alpha} = \frac{\partial x^\mu}{\partial y^\alpha}$$

$$\underbrace{\gamma^{\gamma\beta} \frac{\partial x^\mu}{\partial y^\gamma} \frac{\partial x^\kappa}{\partial y^\beta} \eta_{\kappa\nu} \frac{\partial x^\nu}{\partial y^\alpha}}_{\gamma_{\beta\alpha}} = \delta^\gamma{}_\alpha \frac{\partial x^\mu}{\partial y^\gamma} = \frac{\partial x^\alpha}{\partial y^\alpha} \checkmark$$

$\delta^\gamma{}_\alpha$

$$\bar{P}^\mu{}_\nu = \left(\delta^\mu{}_\nu - P^\mu{}_\nu \right) =$$

$$\bar{P}^\mu{}_\nu \bar{P}^\nu{}_\kappa = \delta^\mu{}_\kappa - 2 P^\mu{}_\kappa + \underbrace{P^\mu{}_\nu P^\nu{}_\kappa}_{P^\mu{}_\kappa}$$

$$= \delta^\mu{}_\kappa - P^\mu{}_\kappa = \bar{P}^\mu{}_\kappa$$

$$\mathrm{Tr}\, \bar{P}^\mu{}_\nu = \delta^\nu{}_\mu \bar{P}^\mu{}_\nu = 3 - \delta^\nu{}_\mu P^\mu{}_\nu$$

$$= 1$$

So it's a 1-dim projection operator:

$$\bar{P}^{\mu}{}_{\kappa} = v^{\mu} v_{\kappa} \equiv n^{\mu} n_{\kappa}$$

$$\bar{P}^{\mu}{}_{\kappa} n^{\kappa} = n^{\mu}$$

$$R_{\alpha\beta\gamma\delta} = -\left(M_{\alpha\gamma} M_{\beta\delta} - M_{\alpha\delta} M_{\beta\gamma} \right)$$