exercise 6:

problem 1:

$$\nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$$
$$+ \Gamma^\lambda{}_{\mu\nu} A_\lambda - \Gamma^\lambda{}_{\nu\mu} A_\lambda$$
$$= \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\nabla_\mu F_{\alpha\beta} + \nabla_\beta F_{\mu\alpha} + \nabla_\alpha F_{\beta\mu} = \partial F + 2\,\text{perm}$$
$$+ \Gamma^\lambda{}_{\alpha\mu} F_{\lambda\beta} + \Gamma^\lambda{}_{\beta\mu} F_{\alpha\lambda}$$
$$+ \Gamma^\lambda{}_{\mu\beta} F_{\lambda\alpha} + \Gamma^\lambda{}_{\beta\gamma} F_{\mu\lambda}$$
$$+ \Gamma^\lambda{}_{\alpha\beta} F_{\lambda\mu} + \Gamma^\lambda{}_{\alpha\mu} F_{\beta\lambda}$$

$$= \partial_\mu F_{\alpha\beta} + \partial_\beta F_{\mu\alpha} + \partial_\alpha F_{\beta\mu}$$

problem 2:

$$ds^2 = (d\theta^2 + \cos^2\theta \, d\varphi^2) L^2$$

$$x^\mu = \begin{pmatrix} \theta \\ \varphi \end{pmatrix} \qquad g_{\mu\nu} = \begin{pmatrix} 1 & \\ & \cos^2\theta \end{pmatrix} L^2$$

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda k} \left\{ \frac{\partial g_{\mu k}}{\partial x^\nu} + \frac{\partial g_{\nu k}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^k} \right\}$$

The only non-vanishing derivative of the metric is

$$\frac{\partial g_{\varphi\varphi}}{\partial \theta} = \frac{\partial \cos^2\theta}{\partial \theta} L^2$$

Hence:

$$\Gamma^\theta{}_{\varphi\varphi} = - \frac{1}{2} \frac{\partial g_{\varphi\varphi}}{\partial \theta} \quad \frac{1}{L^2}$$

$$\Gamma^\varphi{}_{\varphi\theta} = \Gamma^\varphi{}_{\theta\varphi} = \frac{1}{2} \frac{1}{\cos^2\theta} \quad \frac{\partial g_{\varphi\varphi}}{\partial \theta} \frac{1}{L^2}$$

And all others vanish.

The Riemann tensor is given by

$$R_{\mu\nu\lambda\kappa} = \frac{1}{2}\left(\frac{\partial^2 g_{\mu\lambda}}{\partial x^\nu \partial x^\kappa} - \frac{\partial^2 g_{\mu\kappa}}{\partial x^\lambda \partial x^\nu} + \frac{\partial^2 g_{\nu\kappa}}{\partial x^\lambda \partial x^\mu} - \frac{\partial^2 g_{\nu\lambda}}{\partial x^\kappa \partial x^\mu} \right)$$

$$+ g_{\rho\sigma} \left\{ \Gamma^\rho_{\mu\lambda} \Gamma^\sigma_{\nu\kappa} - \Gamma^\rho_{\mu\kappa} \Gamma^\sigma_{\nu\lambda} \right\}$$

In 2D the only non-vanishing elements are

$$R_{\theta\varphi\theta\varphi} = R_{\varphi\theta\varphi\theta} = -R_{\theta\varphi\varphi\theta} = -R_{\varphi\theta\theta\varphi}$$

$$R_{\varphi\theta\varphi\theta} = \frac{1}{2} \frac{\partial^2 g_{\varphi\varphi}}{\partial\theta\partial\theta}$$

$$g_{\varphi\varphi} \left\{ \Gamma^\varphi_{\varphi\varphi} \Gamma^\varphi_{\theta\theta} - \Gamma^\varphi_{\varphi\theta} \Gamma^\varphi_{\varphi\theta} \right\}$$

$$= \frac{1}{2} \frac{\partial^2 g_{\varphi\varphi}}{\partial\theta\partial\theta} - g^{\varphi\varphi} \frac{1}{4} \left(\frac{\partial g_{\varphi\varphi}}{\partial\theta} \right)^2 \frac{1}{L^4}$$

$$= L^2 \left(\sin^2\theta - \cos^2\theta - \sin^2 \right) = -\cos^2 L^2$$

$$R = g^{\lambda\mu} g^{\nu\rho} R_{\lambda\nu\mu\rho} = 2 g^{\varphi\varphi} g^{\theta\theta} R_{\varphi\theta\varphi\theta}$$

$$= -2/L^2$$

Problem 3+4: D dimensions

$$R_{\mu\nu\lambda\kappa} = A\left(g_{\mu\lambda} g_{\nu\kappa} - g_{\mu\kappa} g_{\nu\lambda}\right) R$$
$$+ B\left(g_{\mu\lambda} R_{\nu\kappa} - g_{\mu\kappa} R_{\nu\lambda}\right.$$
$$\left. + g_{\nu\kappa} R_{\mu\lambda} - g_{\nu\lambda} R_{\mu\kappa}\right)$$
$$+ C_{\mu\nu\lambda\kappa}$$

$$g^{\mu\lambda} R_{\mu\nu\lambda\kappa} = R_{\nu\kappa} =$$

$$B\left(D R_{\nu\kappa} - R_{\nu\kappa} + g_{\nu\kappa} R - R_{\nu\kappa}\right)$$
$$+ A\left(D g_{\nu\kappa} - g_{\nu\kappa}\right) R$$

$$= R_{\nu\kappa}\left(BD - 2B\right) + R\left(B + AD - A\right)$$

$$B = \frac{1}{D-2} \qquad A = \frac{-B}{(D-1)}$$
$$= -\frac{1}{(D-1)} \frac{1}{(D-2)}$$

D	A	B
3	$-\frac{1}{2}$	1
4	$-\frac{1}{6}$	$\frac{1}{2}$