solutions - exercises 2:

problem 1:

We can go to a frame where $U^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$U^\mu U_\mu = V^\mu V_\mu = -1$$

In this frame V^μ will be of the form

$$V^\mu = \gamma \begin{pmatrix} 1 \\ \vec{v} \end{pmatrix}$$

$$\Rightarrow \quad U^\mu V_\mu = -\gamma = -\sqrt{\frac{1}{1-\vec{v}^2}}$$

$$\Rightarrow \quad |v| = \sqrt{1 - \frac{1}{(U^\mu V_\mu)^2}}$$

problem 2:

The product of the two boosts is given by

$$\Lambda^\kappa{}_\mu(\varphi_1)\,\Lambda^\mu{}_\lambda(\varphi_2)$$

$$= \begin{pmatrix} A & B & & \\ B & A & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$A = \cosh\varphi_1 \cosh\varphi_2 + \sinh\varphi_1 \sinh\varphi_2 = \cosh(\varphi_1+\varphi_2)$$

$$B = \cosh\varphi_1 \sinh\varphi_2 + \cosh\varphi_2 \sinh\varphi_1 = \sinh(\varphi_1+\varphi_2)$$

$$\Lambda^\mu{}_\kappa(\varphi_1)\Lambda^\kappa{}_\nu(\varphi_2) = \Lambda^\mu{}_\nu(\varphi_1+\varphi_2)$$

problem 3:

One can write the rotation as

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & \\ & R_{ij} \end{pmatrix}^\mu{}_\nu$$

And the boost along x as

$$\Lambda^x = \begin{pmatrix} c & -s & & \\ -s & c & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

In order to generate a general boost, one can rotate + boost + rotate back

$$\begin{pmatrix} 1 & \\ & R \end{pmatrix}\left(\Lambda^x\right)\begin{pmatrix} 1 & \\ & R^T \end{pmatrix}$$

The genral boost is given by

$$\Lambda = \begin{pmatrix} \gamma & -\gamma v_i \\ -\gamma v_i & \delta_{ij} + (\gamma-1)\frac{v_i v_j}{v^2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & \\ & R \end{pmatrix}\Lambda\begin{pmatrix} 1 & \\ & R^T \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\gamma (Rv)_i \\ -\gamma (R\cdot v)_i & R\delta R^T + (\gamma-1)\frac{(Rv)_i (Rv)_j}{v^2} \end{pmatrix}$$

$$R\delta R^T = \delta_{ij}$$

The final result is that
rotation+boost+rotation back
is equivalent to a boost with a rotated velocity.

problem 4:

The proof follows the same proof for the energy momentum tensor in the lecture.

$$j^\mu = \sum q_n \; \delta^3(x - x_n) \frac{dx_n^\mu}{dt}$$

$$\partial_i j^i = \sum q_n \partial_i \delta(x - x_n) \frac{dx_n^i}{\partial t}$$

$$= \sum q_n \left(-\frac{\partial}{\partial x_n}\right) \delta(x - x_n) \frac{dx_n^i}{dt}$$

$$= \sum q_n (-\partial_t) \delta(x - x_n)$$

$$= -\partial_t j^0 \Rightarrow \partial_\mu j^\mu = 0$$

problem 5:

$$\partial_\mu j^\mu = 0 \Rightarrow$$

$$0 = \int d^3x \; \partial_\mu j^\mu = \int d^3x \; (\partial_t j^0) + \underbrace{\int d^3x \; \vec{\nabla} \vec{j}}$$

boundary term = 0
(by assumption)

$$\rightarrow \quad 0 = \partial_t \int d^3x\, j^0 = \partial_t Q$$

problem 6:

$$T^{\mu\nu}_{em} = F^{\mu}{}_{\alpha} F^{\nu\alpha} - \frac{1}{4}\eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$$

$$\partial_\mu T^{\mu\nu}_{em} = (\partial_\mu F^{\mu}{}_{\alpha}) F^{\nu\alpha} + F^{\mu}{}_{\alpha} \partial_\mu F^{\nu\alpha} - \frac{1}{2}\eta^{\mu\nu}(\partial_\mu F_{\alpha\beta}) F^{\alpha\beta}$$

Bianchi: $\partial_\mu F_{\alpha\beta} + \partial_\beta F_{\mu\alpha} + \partial_\alpha F_{\beta\mu} = 0$

$$F^{\alpha\beta}(\partial_\mu F_{\alpha\beta} + \partial_\beta F_{\mu\alpha} + \partial_\alpha F_{\beta\mu})$$

$$= F^{\alpha\beta}(\partial_\mu F_{\alpha\beta} + \partial_\beta F_{\mu\alpha} - \partial_\beta F_{\alpha\mu})$$

$$= F^{\alpha\beta}(\partial_\mu F_{\alpha\beta} + 2\partial_\beta F_{\mu\alpha})$$

$$\Rightarrow \partial_\mu T^{\mu\nu}_{em} = - j_\alpha F^{\nu\alpha}$$

problem 7:

First, we show the following relation

$$R^{ia} R^{jb} R^{kc} \epsilon^{abc} = \det R\, \epsilon^{ijk}$$

First observe that the left-hand side is completly antisymmetric in the indices ijk.

So

$$R R R \epsilon \propto \epsilon$$

The proportionality is given by choosing ijk = 123:

$$R^{1a} R^{2b} R^{3c} \epsilon^{abc} = \det R$$

If R is a rotation matrix then

$$\det R = 1 \; ; \qquad R^{ai} R^{bi} = \delta^{ab}$$

$$R^{ia} R^{jb} \underbrace{R^{kd} R^{kc}}_{\delta^{dc}} \epsilon^{abc} = R^{kd} \epsilon^{ijk}$$

$$R^{ia} R^{jb} \epsilon^{abd} = R^{kd} \epsilon^{ijk}$$

The tensor components F^{ij} transform under the spatial rotations as

$$F^{ij} \to R^{ia} R^{jb} F^{ab} = R^{ia} R^{jb} \epsilon^{ijk} B^{k}$$

$$= R^{kd} \epsilon^{ijk} B^{d}$$

$$= \epsilon^{ijk} (R^{kd} B^{d})$$

$$B^{d} \to R^{kd} B^{d} = R \cdot B$$