Exercises, week 8

1 S^2

Calculate the metric of S^2 in R^3, i.e. the induced metric of the embedding

$$x^\mu(\theta, \phi) = L \begin{pmatrix} \sin\theta \sin\phi \\ \sin\theta \cos\phi \\ \cos\theta \end{pmatrix} . \tag{1}$$

into flat Euclidean space $g_{\mu\nu} = \eta_{\mu\nu}$. Also determine the line element ds^2.

2 Gauss curvature, explicit

Consider the embedding given by a function f

$$x^\mu = \begin{pmatrix} X \\ Y \\ f(X,Y) \end{pmatrix}, \quad y^a = \begin{pmatrix} X \\ Y \end{pmatrix}. \tag{2}$$

Calculate explicitly the first fundamental form (=induced metric) that is given by ($\eta_{\mu\nu}$ is the Euclidean 3x3 matrix)

$$\gamma_{ab} = \frac{dx^\mu}{dy^a}\frac{dx^\nu}{dy^b}\eta_{\mu\nu} \tag{3}$$

and the second fundamental form that is given by

$$M_{ab} = \frac{d^2 x^\mu}{dy^a dy^b} n_\mu \tag{4}$$

with the normal given by

$$n_\mu = \epsilon_{\mu\nu\lambda}\frac{dx^\nu}{dy^a}\frac{dx^\lambda}{dy^b}\frac{\epsilon^{ab}}{2\sqrt{det\gamma}} . \tag{5}$$

Show that n_μ is properly normalized, $n_\mu n^\mu = 1$.

[These quantities provide a link between the Gauss curvature and the Riemann tensor:

$$-\frac{R}{2} = \frac{\mathrm{detM}}{\det\gamma} = K . \tag{6}$$

where K is the Gauss curvature, M is the second fundamental form and γ is the induced metric, also called first fundamental form. R is the Ricci scalar derived from the induced metric (you don't have to show this relation).]