Exercises, week 6

1 Covariant Bianchi identity

Show that for the gauge potential A_μ the field strenght in GR reduces to the ordinary definition

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1}$$

and that the Bianchi identity reduces to the ordinary definition

$$\nabla_\alpha F_{\mu\nu} + \nabla_\nu F_{\alpha\mu} + \nabla_\mu F_{\nu\alpha} = \partial_\alpha F_{\mu\nu} + \partial_\nu F_{\alpha\mu} + \partial_\mu F_{\nu\alpha} \tag{2}$$

2 Sphere in 2D

Consider a space with Euclidean signature and 2D line element of a sphere

$$ds^2 = L^2(d\theta^2 + \cos^2(\theta)d\phi^2)\,. \tag{3}$$

Determine the non-vanishing entries of the Christoffel symbol, the Riemann tensor $R^\mu{}_{\nu\kappa\lambda}$, the Ricci tensor $R_{\nu\lambda} = R^\mu{}_{\nu\mu\lambda}$ and the curvature scalar $R = R^\mu{}_{\nu\mu\lambda} g^{\nu\lambda}$.

3 Riemann tensor - decomposition in 3D

We have seen that in 2D the Riemann tensor is related to the curvature scalar as

$$R_{\mu\nu\kappa\lambda} = \frac{1}{2}(g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa})\,R \tag{4}$$

In 3D, the Riemann tensor has 6 degrees of freedom and it is tempting to express the Riemann tensor in terms of the Ricci tensor (that also has 6 degrees of freedom in 3D). In fact, this is possible and the Riemann tensor can be written as a linear combination of

$$(g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa})\,R \tag{5}$$

and

$$(g_{\mu\kappa}R_{\nu\lambda} - g_{\mu\lambda}R_{\nu\kappa} + g_{\nu\lambda}R_{\mu\kappa} - g_{\nu\kappa}R_{\mu\lambda})\,. \tag{6}$$

Check that these terms have the right symmetries (symmetric under $12 \leftrightarrow 34$, antisymmetric under $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ and cyclic under permutation of 234).

Fix the proportionality coefficients (in 3D) by the requirement that $R_{\mu\nu}$ and R are correctly reproduced.

4 Riemann tensor - decomposition in 4D

In 4D, the Riemann tensor contains 20 degrees of freedom, while $R_{\mu\nu}$ contains 10. Still, one can decompose the Riemann tensor in terms of the form (5) and (6) and a remainder $C_{\mu\nu\kappa\lambda}$ that fulfills

$$g^{\mu\kappa} C_{\mu\nu\kappa\lambda} = 0. \tag{7}$$

What are the proportionality coefficients (in 4D)? How many degrees of freedom does C have?