Exercises, week 4

1 Christoffel symbol

Using the definition of the Christoffel symbol

$$\left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} = \frac{1}{2} g^{\lambda\rho} \left[\frac{dg_{\rho\nu}}{dx^\mu} + \frac{dg_{\rho\mu}}{dx^\nu} - \frac{dg_{\mu\nu}}{dx^\rho} \right] \tag{1}$$

show that it transforms as an affine connection, i.e.

$$\left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\}' = \frac{dx'^\lambda}{dx^\alpha} \frac{dx^\beta}{dx'^\mu} \frac{dx^\gamma}{dx'^\nu} \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} + \frac{dx'^\lambda}{dx^\rho} \frac{d^2 x^\rho}{dx'^\mu dx'^\nu} \,. \tag{2}$$

2 Particle motion

Show that the equation of motion for a particle in a gravitational field

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0\,. \tag{3}$$

transforms as a contravariant vector using the transformation rule of an affine connection Γ.

3 Covariant derivative 1

Show that the covariant derivative of a contravariant vector

$$\nabla_\kappa V^\mu = \partial_\kappa V^\mu + \Gamma^\mu_{\kappa\lambda} V^\lambda\,, \tag{4}$$

transforms as a tensor using the transformation rule of an affine connection Γ.

4 Covariant derivative 2

Show that the covariant derivative of a covariant vector

$$\nabla_\kappa V_\mu = \partial_\kappa V_\mu - \Gamma^\lambda_{\kappa\mu} V_\lambda \,, \tag{5}$$

transforms as a tensor using the transformation rule of an affine connection Γ.

5 Covariant derivative 2

Show that the covariant derivative of the metric vanishes

$$\nabla_\kappa g_{\mu\nu} = 0, \tag{6}$$

in case the affine connection is given by the Christoffel symbol, $\Gamma^\lambda_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\}$.