Exercises, week 3

1 Sylvester's law of inertia

The metric transforms under a coordinate transformation as

$$g'_{\mu\nu} = g_{\alpha\beta}\Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}. \tag{1}$$

where $\Lambda^{\alpha}{}_{\mu} = \frac{dx^{\alpha}}{dx'^{\mu}}$.

Show that using the appropriate (real) Λ, the matrix g' can be made diagonal with only $1, -1, 0$ on the diagonal (assuming g is symmetric). Also show that the number of occurences of $1, -1, 0$ on the diagonal is independent from the choice of Λ. Notice that the order of the elements is not fixed and the transformation Λ is typically not unique.

2 Geodesic 1

Show that the line element along a geodesic is constant, i.e.

$$\frac{d}{d\lambda}\left(g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}\right) = 0\,, \tag{2}$$

using the geodesic equation for x^{μ} and the definition of the Christoffel symbol Γ in terms of g.

3 Geodesic 2

Show that the Euler-Lagrange equation of the following "energy functional" leads to the geodesic equations

$$S_E = \frac{1}{2}\int d\lambda\left(-g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}\right)\,. \tag{3}$$

4 Geodesic 3

The proper length of a path is given by the action

$$S_l = \int d\lambda \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \,. \tag{4}$$

Show that this action is invariant under reparameterizations $\lambda \to \lambda' = f(\lambda)$. Derive the Euler-Lagrange equations and show that they are invariant under reparameterizations.

Show that extrema of the energy functional (3) also extremize the path length (4). Also show that the opposite is not true.