Exercises, week 2

1 Relative velocity

Since velocities do not behave additively in special relativity, there is the question how to determine the relative velocities between two objects in a frame-independent way. Given two velocities U^μ and V^μ, one way of defining this is using

$$\Delta v = \sqrt{1 - \frac{1}{(U_\mu V^\mu)^2}}. \tag{1}$$

Derive this expression by boosting into the frame where one of the objects is at rest.

2 Rapidities

We have seen that the Lorentz-transformation for a boost in the x-direction is of the form

$$\Lambda^\mu{}_\nu(\phi) = \begin{pmatrix} \cosh(\phi) & -\sinh(\phi) & 0 & 0 \\ -\sinh(\phi) & \cosh(\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

Show that the rapidity ϕ indeed behaves additively by calculating $\Lambda^\mu{}_\nu(\phi_1)\Lambda^\nu{}_\kappa(\phi_2)$.

3 General boosts

Show that boosts in a general direction are given by

$$\Lambda^\mu{}_\nu(\phi) = \begin{pmatrix} \gamma & -\gamma\vec{v}_i \\ -\gamma\vec{v}_j & \delta_{ij} + (\gamma - 1)\vec{v}_i\vec{v}_j/v^2 \end{pmatrix} \tag{3}$$

by rotating a boost in the x-direction into other directions.

4 Current conservation

Show that the following current is conserved (q_n is the charge for the particle n)

$$J^\mu = \sum_n q_n \delta(\vec{x} - \vec{x}_n) \frac{dx_n^\mu}{dt} \,. \tag{4}$$

5 Charge conservation

Show that current conservation $\partial_\mu J^\mu$ implies that the following charge is time-independent (assume that the current falls off quickly enough at infinity)

$$Q = \int d^3x \, J^0 \,. \tag{5}$$

6 Electromagnetic energy-momentum tensor

Show that the following tensor involving the field strength $F^{\mu\nu}$

$$T_{em}^{\mu\nu} = F^\mu_{\ \alpha} F^{\nu\alpha} - \frac{1}{4}\eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \tag{6}$$

has as the divergence

$$\partial_\mu T_{em}^{\mu\nu} = -F^\nu_{\ \gamma} J^\gamma \,, \tag{7}$$

by writing the field strength tensor in terms of the potential A^μ and using Maxwell's equations (you also can use the Bianchi identity from last week to simplify this derivation).

7 Magnetic field

In special relativity, the spatial coordinates of the field stregth tensor are given by the magnetic field

$$F^{ij} = \epsilon^{ijk} B_k \tag{8}$$

Show that B_k transforms as 3-vector under normal spatial roations even though F^{ij} transform as an antisymmetric tensor.