tensors

We study the behavior of several quantities under coordinate transformations.

Consider a contravariant vector: $x \to x'$

$$V'^{\mu} = \underbrace{\frac{dx'^{\mu}}{dx^{\nu}}}_{\Lambda^{\mu}{}_{\nu}(x)} V^{\nu}$$

with the prototypical example the differential:

$$dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} dx^{\nu}$$

Notice that in SR, the coordinate itself was a contravariant vector:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

In GR it is not: $x'^{\mu} = f(x^{\nu})$

Consider a covariant vector:

$$V'_{\mu} = \frac{dx^{\nu}}{dx'^{\mu}} V_{\nu}$$

with the prototypical example
the derivative acting on a scalar:

$$\left(\frac{d}{dx^\mu}\varphi(x) \equiv \partial_\mu \varphi(x)\right)' = \frac{dx^\nu}{dx'^\mu}\partial_\nu \varphi(x)$$

In general, a tensor

$$\left(T^{\mu\nu}{}_\alpha\right)' = \frac{dx'^\mu}{dx^\lambda}\frac{dx'^\nu}{dx^\kappa}\frac{dx^\beta}{dx'^\alpha} T^{\lambda\kappa}{}_\beta$$

with the prototype the metric

$$g'_{\mu\nu} = \frac{dx^\alpha dx^\beta}{dx'^\mu dx'^\nu} g_{\alpha\beta}$$

in order to have a invariant line element

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

$$= g'_{\mu\nu}\, dx'^\mu dx'^\nu = ds'^2$$

Notice that

$$\frac{dx^\mu}{dx'^\nu}\frac{dx'^\nu}{dx^\lambda} = \delta^\mu{}_\lambda$$

Contractions can be used to form
scalars:

$$T^{\lambda\mu} g_{\mu\lambda} \equiv T^\lambda{}_\lambda$$

The metric can also be used to lower/raise indices:

$$T^{\mu}{}_{\nu} \equiv T^{\mu\alpha} g_{\alpha\nu}$$

$$\left(T^{\mu\alpha} g_{\alpha\nu}\right)' = \frac{dx'^{\mu}}{dx^{\lambda}} \underbrace{\frac{dx'^{\alpha}}{dx^{\beta}} \frac{dx^{k}}{dx'^{\alpha}}}_{\delta^{k}_{\beta}} \frac{dx^{\sigma}}{dx'^{\nu}} T^{\lambda\beta} g_{k\sigma}$$

$$= \frac{dx'^{\mu}}{dx^{\lambda}} \frac{dx^{\sigma}}{dx'^{\nu}} T^{\lambda\beta}{}_{\beta\sigma} =$$

$$(\qquad) \; T^{\lambda}{}_{\sigma}$$

$$\delta^{\mu}{}_{\alpha} \Rightarrow \underbrace{\delta_{\mu\alpha}}_{\text{not used}} = g_{\mu\nu} \delta^{\nu}{}_{\alpha} \overset{?}{=} \mathbb{1}$$

$$= g_{\mu\alpha}$$

So $\delta^{\mu}{}_{\nu}$, $\delta_{\mu}{}^{\nu}$, $g_{\mu\nu}$, $g^{\mu\nu}$

are related by raising/lowering indices.

Notice that the $\delta^{\mu}{}_{\nu}$ is a mixed tensor with constant elements.

$$\left(\delta^{\mu}{}_{\nu}\right)' = \frac{dx'^{\mu}}{dx^{\lambda}} \frac{dx^{k}}{dx'^{\nu}} \delta^{\lambda}{}_{k}$$

$$= \frac{dx'^{\mu}}{dx^{\lambda}} \frac{dx^{\lambda}}{dx'^{\nu}} = \delta^{\mu}{}_{\nu}$$

One infamous exception to this notation is the affine connection $\Gamma^{\mu}{}_{\nu\lambda}$ which is not a tensor, but a 'symbol'

Remember: the goal is to make out of all physical laws tensor relations
-> we can deduce them in any coordinate system.

tensor algebra:

A) linear combinations:

$$T^{\mu}{}_{\nu} = \alpha A^{\mu}{}_{\nu} + \beta B^{\mu}{}_{\nu}$$

B) direct product

$$T^{\mu}{}_{\nu} = A^{\mu} B_{\nu}$$

$$(T^{\mu}{}_{\nu})' = (A^{\mu})' B_{\nu}' = \frac{dx'^{\mu}}{dx^{\alpha}} \frac{dx^{\beta}}{dx'^{\nu}} A^{\alpha} B_{\beta}$$

$$= \text{---} \, ' \, \text{---} \quad T^{\mu}{}_{\nu}$$

C) contraction:

$$F^{\mu}{}_{\rho} \equiv T^{\mu\nu}{}_{\rho\nu}$$

So most analysis of SR carries over to the tensor analysis of GR. Notable differences are: derivatives of tensors, Levi-Civita symbol

tensor densities:

One important class of non-tensors are tensor densities. Consider

$$g = -\det g_{\mu\nu}$$

$$g' = -\det g'_{\alpha\nu} = -\left|\frac{dx}{dx'}\right|^2 \det g_{\mu\nu}$$
$$= \left|\frac{dx}{dx'}\right|^2 g$$

With $\left|\frac{dx'}{dx}\right| = \det \frac{dx'^\mu}{dx^\nu}$ is the

Jabobian of the coordinate transformation

$x \to x'$.

A quantity as g is called a scalar density, since it transforms as a scalar up to factors of the Jacobian.

Tensor densities are defined accordingly: They transform as tensors up to the Jacobian. The weight of a tensor density is determined by the number of factors $\left|\frac{dx'}{dx}\right|$.

For example: g is a scalar density of weight -2.

Any tensor density can be made a tensor by multiplying with an appropriate factor $g^{+w/2}$

For example the integration measure d^4x

$$d^4x' = \left|\left|\frac{dx'}{dx}\right|\right| d^4x$$

has weight 1.
This means

$$d^4x \sqrt{g}$$

is a scalar.

Levi-Civita symbol

We define

$$\epsilon^{\mu\nu\kappa\lambda} \equiv \begin{cases} 1 & \text{even permutations of} \\ -1 & \text{odd} \quad .. \quad 0123 \\ 0 & \text{otherwise} \end{cases}$$

Is this a tensor in GR?

$$(\epsilon^{\mu\nu\kappa\lambda})' \overset{!}{=} \frac{dx'^{\mu} dx'^{\nu} dx'^{\kappa} dx'^{\lambda}}{dx^{\alpha} dx^{\beta} dx^{\gamma} dx^{\delta}} \epsilon^{\alpha\beta\gamma\delta} \quad .$$

$$= \det \frac{dx'}{dx} \quad \epsilon^{\mu\nu\kappa\lambda}$$

$$= \left| \frac{dx'}{dx} \right| \epsilon^{\mu\nu\kappa\lambda}$$

So $\epsilon^{\mu\nu\kappa\lambda}$ is a tensor density with weight -1.

This means that

$$\epsilon^{\mu\nu\kappa\lambda} / \sqrt{g}$$

is a tensor.
What about

not a tensor relation.

$$\epsilon_{\mu\nu\lambda\kappa} = g_{\alpha\mu} g_{\beta\nu} g_{\gamma\lambda} g_{\delta\kappa} \quad \epsilon^{\alpha\beta\gamma\delta} = -g\, \epsilon^{\mu\nu\lambda\kappa}$$

This also has weight -1, so a tensor is

$$\epsilon_{\mu\nu\lambda\kappa} / \sqrt{g}$$

but $\epsilon_{\mu\nu\lambda\kappa}$ does not only involve 1,-1,0.