The Newton limit

In order to make contact with Newton's theory, we consider a particle that is moving slowly (non-relativistically) in a stationary gravitational backround.

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta}\frac{\partial x^\alpha}{\partial \tau}\frac{\partial x^\beta}{\partial \tau} = 0$$

$$\frac{\partial x^\alpha}{\partial \tau} = \frac{dt}{d\tau}\begin{pmatrix}1\\ \vec{v}\end{pmatrix} \qquad \vec{v} \ll 1$$

$$\simeq \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\frac{dt}{d\tau}$$

$$\frac{d^2x^\mu}{\partial \tau^2} + \Gamma^\mu{}_{00}\left(\frac{\partial t}{\partial \tau}\right)^2 = 0 \qquad \text{stationary}$$

$$\downarrow$$

$$\Gamma^\mu{}_{00} \simeq -\frac{1}{2} g^{\mu\nu}\frac{\partial g_{00}}{\partial x^\nu} \qquad \left(\frac{\partial g_{\mu\nu}}{\partial t} = 0\right)$$

If gravitational effects are weak, the metric should be close to the Minkowski metric

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \qquad (h \ll 1)$$

$$\Gamma^\mu{}_{00} = -\frac{1}{2}\eta^{\mu\nu}\frac{\partial h_{00}}{\partial x^\nu}$$

$$\hookrightarrow \quad \frac{d^2\vec{x}}{d\tau^2} = \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2 \vec{\nabla} h_{00}$$

$$\frac{d^2t}{d\tau^2} = 0 \rightarrow \quad t = c\tau + a$$

$$\frac{d^2\vec{x}}{dt^2} = \frac{1}{2}\vec{\nabla} h_{00} = -\vec{\nabla}\Phi$$

$$h_{00} = -2\Phi + \text{const}$$

So the 00 component of the metric has to contain the gravitational potential:

$$g_{00} = -(1 + 2\Phi)$$

and Φ is the gravitational potential as in the Newton force, e.g.

$$\Phi = -\frac{GM}{r}$$

for a heavy central object.

Time dilatation revisited:

The proper time in the free falling frame is given by

$$d\tau^2 = -ds^2 = -\eta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta$$

and in general

$$d\tau^2 = -ds^2 = -g_{\mu\nu}\, dx^\mu dx^\nu$$

So the time dilatation is given by

$$\left(\frac{d\tau}{dt}\right)^2 = -g_{\mu\nu} \frac{dx^\mu}{dt}\frac{dx^\nu}{dt} \qquad x^\mu = \begin{pmatrix} t \\ \vec{x} \end{pmatrix}$$

$$\simeq \underbrace{(1+2\Phi)}_{\text{GR time dilatation from grav. pot.}} - \underbrace{\vec{v}^2}_{\text{Special relativistic time dilatation}} \qquad \vec{v} = \frac{d\vec{x}}{dt}$$

So if a light source produces a monochromatic light and it is observed at another place, the observed frequency will shift

$$\frac{\Delta\nu}{\nu} = \Phi(x_2) - \Phi(x_1)$$

[remember Pound/Rebka and Moessbauer]

Interpretation with the weak E.P.

The photon of frequency ν carries an energy and a mass $\Delta m_1 = -h\nu_1$

at the same time, the observed photon carries mass and energy

$$\Delta m_2 = h\nu_2$$

But the photon also carries gravitational energy from the gravitational potential.

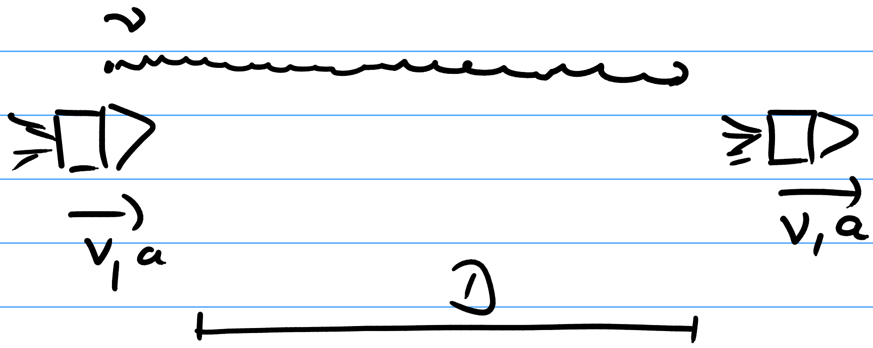
The weak E.P. demands that after the absorption the mass of the combined system does not change

$$\hookrightarrow \quad 0 = \Delta m_1 (1 + \Phi_1) + \Delta m_2 (1 + \Phi_2)$$

$$- \frac{\Delta m_2}{\Delta m_1} = \frac{\nu_2}{\nu_1} = \frac{1 + \Phi_1}{1 + \Phi_2} \approx 1 + \Phi_1 - \Phi_2$$

Example with rockets:

Imagine 2 rockets with a velocity v, accelerating in tandem and sending light signals

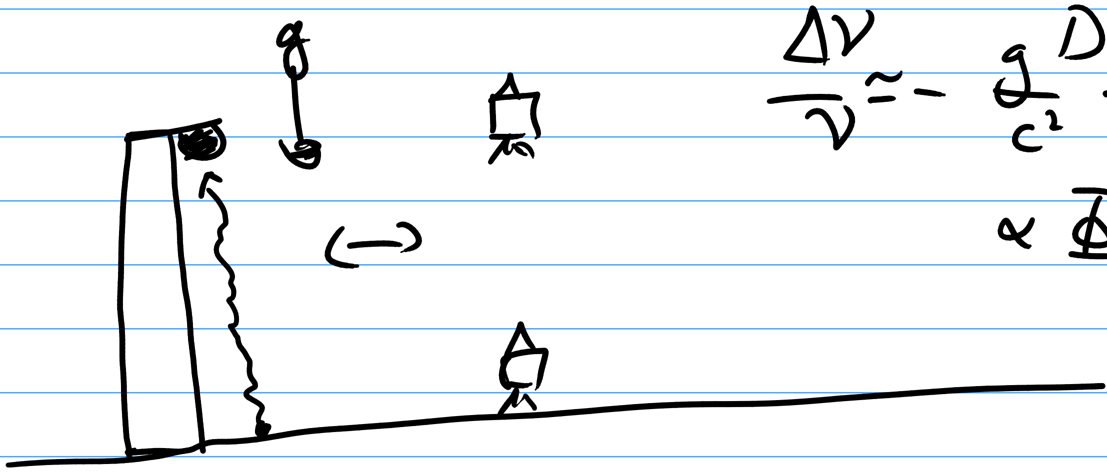


$$\frac{\Delta\nu}{\nu} \simeq -\frac{\Delta v}{c} \simeq -\frac{Da}{c^2}$$

Now consider a tower on the earth's surface. If a signal is sent from the base to the top, the situation should be equivalent to the two rockets with a = g .

$$\vec{\nabla}\underline{\Phi} = \vec{g}$$

$$\frac{\Delta\nu}{\nu} \simeq -\frac{gD}{c^2} = \propto \underline{\Phi}_1 - \underline{\Phi}_2$$



Tensor analysis

So far, we used the Principle of Equivalence:
There is a free falling frame where gravity
is absent and the other laws of physics are SR.

The laws in a general frame can then be
deduced via trasforming to the free falling
frame.

One way of simplifying this step, is by
formulating all laws in a general covariant
way -> Principle of general covariance.

Notice that there is a difference between
special relativity and general relativity:
Special relativity is a symmetry. In General
relativity, the metric changes by the
transformations and the "symmetry" tells how
gravity will couple/act in different contexts.