General theory of relativity

The starting point of general relativity is the equivalence principle. It is based on the observation that the 'gravitational mass' and the 'inertial mass' are experimentally the same

Newton's law: $$m_I \vec{a} = m_I \frac{d^2\vec{x}}{dt^2} = \vec{F}$$

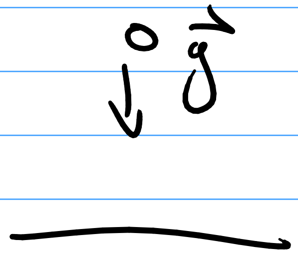
gravitational force: $$\vec{F} = -m_g \vec{\nabla}\Phi$$

$$\hookrightarrow \vec{a} = -\vec{\nabla}\Phi$$

So the acceleration due to gravity is independent from the object.

Hence it is possible to transform (locally) to a accelerated frame where gravitational forces are absent.

E.g.



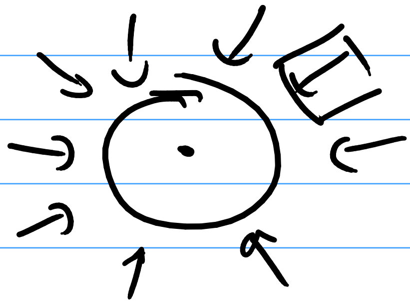
$$\vec{F} = m\vec{a} = m\vec{g}$$

$$\hookrightarrow \vec{x} = \vec{x}_0 + \frac{1}{2}\vec{g}t^2$$

$$\vec{x}\,' = \vec{x} - \frac{1}{2}\vec{g}t^2 + \vec{const}\cdot t$$

$$\hookrightarrow m\vec{a}\,' = 0$$

Obviously, this is only possible locally, since for example no frame will eliminate all gravitational forces around the earth.



There is a strong version and a weak version of the equivalence principle (E.P.). In the strong version all physical laws are the same and gravity is absent in the free falling frame (= frame without gravitational forces). In the weak E.P. only the graviational forces are removed but the other forces might change somehow.

It is hard to imagine a theory that fulfills the weak E.P. but not the strong one:

Consider a H-atom:

$$m_H = m_p + m_e - E_b \quad \text{(inertial mass)}$$

E_b = binding energy

The gravitational mass has to know about the binding energy!

-> Also EM fields couple to gravity.

Gravitational forces (beyond Newton)

Consider a particle moving in its local inertial (=free falling) frame.
-> no gravitational forces

$$\frac{d^2\xi^\mu}{d\tau^2} = 0$$

τ is the proper time $\quad d\tau^2 = -\eta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta$

How would this equation change in a different coordinate system:

$$\xi^\alpha(x^\mu)$$

$$0 = \frac{d}{d\tau}\left(\frac{dx^\mu}{d\tau} \frac{d\xi^\alpha}{\partial x^\mu} \right)$$

$$= \frac{d^2x^\mu}{d\tau^2} \frac{d\xi^\alpha}{\partial x^\mu} + \frac{\partial \xi^\alpha}{\partial x^\mu \partial x^\nu} \frac{\partial x^\mu}{\partial \tau} \frac{\partial x^\nu}{\partial \tau}$$

If the transformation into the new coordinate system can be inverted, we also have

$$\frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial x^\mu}{\partial \xi^\beta} = \delta^\alpha{}_\beta$$

$$0 = \frac{d^2 x^\kappa}{d\tau^2} + \left(\frac{\partial x^\kappa}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \right) \frac{\partial x^\mu}{\partial \tau} \frac{\partial x^\nu}{\partial \tau}$$

$$\equiv \frac{d^2 x^\kappa}{(d\tau)^2} + \Gamma^\kappa_{\mu\nu} \frac{\partial x^\mu}{d\tau} \frac{\partial x^\nu}{d\tau}$$

The symbol Γ is called an affine connection.

This will become the geodesic equation in GR.

Also the proper time has to be transformed:

$$d\tau^2 = -\eta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta = -\eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} dx^\mu dx^\nu$$

$$\equiv -g_{\mu\nu}\, dx^\mu dx^\nu$$

which defines the metric tensor g(x).

One can also follow the opposite direction.
Let's assume that the connection $\Gamma(x)$ and
the metric g(x) are given.

Then, at least locally, we can go to a free falling
frame:

$$\xi^\alpha(x) = a^\alpha + b^\alpha{}_\mu (x^\mu - X^\mu)$$
$$+ c^\alpha_{\mu\nu} (x-X)^\mu (x-X)^\nu$$

And also: $\eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \Big|_{x=x} = g_{\mu\nu}(x)$

$$\frac{\partial \xi^\alpha}{\partial x^\mu}\Big|_{x=X} \approx b^\alpha{}_\mu$$

This problem can be inverted, i.e. for every symmetric $g_{\mu\nu}(x)$, one can find the appropriate $b^\alpha{}_\mu$. It is only unique up to Lorentz trafos.

The vector a^α is unconstrained, and the coefficients $c^\alpha{}_{\mu\nu}$ are given by

$$\Gamma^\lambda{}_{\mu\nu} = \frac{\partial x^\lambda}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}$$

$$c^\alpha{}_{\mu\nu} = \frac{1}{2} b^\alpha{}_\lambda \Gamma^\lambda{}_{\mu\nu}$$

Relating the affine connection to the metric

Even though the metric g and the affine connection are independent in every single spacetime point, they are not when considerend along the trajectory of a free-falling particle.

$$g_{\mu\nu}(x) = \frac{d \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta}$$

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta} + \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial^2 \xi^\beta}{\partial x^\lambda \partial x^\nu} \eta_{\alpha\beta}$$

$$\boxed{\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma^\rho{}_{\lambda\mu}\, g_{\rho\nu} + \Gamma^\rho{}_{\lambda\nu}\, g_{\rho\mu}}$$

This relationship can be inverted:

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} = 2 g_{\kappa\nu} \Gamma^\kappa{}_{\lambda\mu}$$

Defining the inverse metric

$$g^{\nu\sigma} = \eta^{\alpha\beta} \frac{\partial x^\nu}{\partial \xi^\alpha} \frac{\partial x^\sigma}{\partial \xi^\beta}$$

$$\hookrightarrow \quad g^{\sigma\lambda} g_{\lambda\kappa} = \delta^\sigma{}_\kappa$$

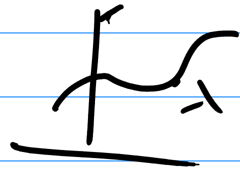
$$\hookrightarrow \quad \Gamma^\sigma{}_{\lambda\mu} = \frac{1}{2} g^{\sigma\nu} \left\{ \frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu} \right\}$$

This is called the Christoffel symbol which forms one possible affine connection.

One special property of the free falling objects is that the line element is constant along the path:

$$\eta, \xi^{\mu}(\lambda) \to x^{\mu}(\lambda), g$$

$$ds^2 = g_{\mu\nu} \, dx^{\mu} dx^{\nu}$$



$$\left(\frac{ds}{d\lambda}\right)^2 = g_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda}$$

$$\frac{d}{d\lambda}\left(\left(\frac{ds}{d\lambda}\right)^2\right) = \frac{d}{d\lambda}\left[g_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda} \right]$$

$$= \frac{\partial g}{\partial x} \frac{\partial x}{\partial \lambda} \left(\frac{\partial x}{\partial \lambda} \frac{\partial x}{\partial \lambda} \right)$$

$$+ g \frac{\partial^2 x}{\partial \lambda^2} \frac{\partial x}{\partial \lambda}$$

↳ geodesic

$$= \underline{\underline{0}}$$

$$\frac{ds}{d\lambda} = \text{const} \qquad \boxed{s \propto \lambda}$$

↳ exercises

Another interesting property is that the path of the free falling object extremizes the integral over the line element:

$ds^2 < 0$

$d\tau^2 = -ds^2$

> 0

$$\ell = \int d\tau$$

$$= \int \sqrt{\underbrace{\underbrace{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}_{\left(\frac{ds}{d\lambda}\right)^2}}_{\left(\frac{d\tau}{d\lambda}\right)^2}}\, d\lambda$$

The solutions of the Euler-Lagrange equations extremize this action.

$\hookrightarrow$ exercises