Every conserved four-current (that also vanishes at infinity) allows to construct a conserved (time-independent) charge

$$Q = \int d^3x \, j^0$$

$$\Rightarrow \quad \partial_t Q = 0$$

[see exercises]

Energy momentum tensor

Instead of the charge, we can construct a four-current of energy and momentum

$$T^{\mu\nu} = \sum_n p_n^\mu \, \delta^3(\vec{x} - \vec{x}_n) \, \frac{dx_n^\nu}{dt}$$

$$= \int d\lambda \sum_n p_n^\mu \, \delta^4(x^\mu - x_n^\mu(\lambda)) \, \frac{dx_n^\nu}{d\lambda}$$

remember: $\quad p^\mu = m \dfrac{dx^\mu}{d\tau}$

$$p^\mu = \begin{pmatrix} E \\ \vec{p} \end{pmatrix} = \gamma m \begin{pmatrix} 1 \\ \vec{v} \end{pmatrix}, \quad \vec{v} = \frac{d\vec{x}}{dt}$$

$$T^{\mu\nu} = \sum_n \frac{p_n^\mu p_n^\nu}{E_n} \, \delta^3(x - x_n(t))$$

So T is symmetric under exchanging the two indices. It is also a tensor.

Is it conserved?

$$\partial_\mu T^{\mu\nu} = 0$$

$$\partial_i T^{i\alpha} = \frac{d}{dx^i} T^{i\alpha}$$

$$= \sum_n p_n^\alpha \frac{dx_n^i}{dt} \frac{d}{dx^i} \delta(\vec{x} - \vec{x}_n(t))$$

$$\text{———} \; n \; \text{————} \left(-\frac{d}{dx_n^i}\right) \delta(\vec{x} - \vec{x}_n(t))$$

$$= \sum_n p_n^\alpha(t) \quad (-\partial_t) \, \delta(\vec{x} - \vec{x}_n(t))$$

$$= -\partial_t \left(\sum_n p_n^\alpha(t) \, \delta(\vec{x} - \vec{x}_n(t)) \right.$$

$$+ \sum_n (\partial_t p_n^\alpha(t)) \, \delta(\vec{x} - \vec{x}_n(t))$$

$$= -\partial_t T^{0\alpha} + G^\alpha$$

$$\partial_\mu T^{\mu\alpha} = G^\alpha$$

with $G^\alpha = \sum_n \delta^3(\vec{x} - \vec{x}_n(t)) \dfrac{dp_n^\alpha}{dt}$

is called the force density.

$$G^\alpha = \sum_n \delta^3(\vec{x} - \vec{x}_n(t)) \frac{d\tau}{dt} f^\alpha$$

In certain situations the force density vanishes. For example, consider local scatterings. Then

$\sum_n p_n^\mu$ does not change in the interaction

For electromagnetic forces, it does not vanish:

$$f_n^\alpha = e_n F^\alpha{}_\gamma \frac{dx_n^\gamma}{dt}$$

$$G^\alpha = F^\alpha{}_\gamma J^\gamma$$

However, we can fix this by adding the electromagentic energy-momentum tensor

$$T_{em}^{\alpha\beta} = F^\alpha{}_\gamma F^{\beta\gamma} - \frac{1}{4}\eta^{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta}$$

Using

$$\partial_\alpha T_{em}^{\alpha\beta} = -F^\alpha{}_\gamma J^\gamma$$

Lagrangian formalism

Consider one degree of freedom as a function of time only $q(t)$

There are two equivalent ways to encode its dynamics

A) Hamiltonian formalism

B) Lagrange formalism

Given some Hamiltonian, e.g.

$$H(q,p) = \frac{1}{2}\frac{p^2}{m} + V(q)$$

The equation of motion (EoM) is given by

$$\dot{q} = \{q, H\} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\dot{p} = \{p, H\} = -\frac{\partial H}{\partial q} = -\frac{\partial V}{\partial q}$$

$$\{A, B\} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}$$

is the Poisson bracket

Alternatively, one can start from the Lagrange density (=Legendre transformation of the Hamiltonian):

$$\mathcal{L}(q,\dot q) = p\dot q - H(p,q)\Big|_{\dot q = \frac{\partial H}{\partial p}}$$

and then solve the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q} = 0$$

Sketch:

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q} \overset{?}{=} 0$$

$$\cancel{\frac{dp}{dq}\dot q} - \frac{\partial H}{\partial q} - \cancel{\frac{\partial H}{\partial p}\frac{\partial p}{\partial q}}$$

$$-\frac{d}{dt}\left[p + \cancel{\dot q\frac{\partial p}{\partial \dot q}} - \cancel{\frac{\partial H}{\partial p}\frac{\partial p}{\partial \dot q}}\right] \overset{?}{=} 0$$

$$-\frac{\partial H}{\partial q} - \dot p \overset{?}{=} 0 \qquad \checkmark$$

The Euler-Lagrange equations imply the Hamiltonian equations. Euler-Lagrange equation is one second order differential equation, while the Hamiltonian equations are two first-order equations.

The Euler-Lagrange equations can be understood to arise from a variational principle:

Consider the action:

$$S = \int dt \, \mathcal{L}(q, \dot{q}, t)$$

and the variations with respect to the degree of freedom

$$q(t) \to q(t) + \delta q(t)$$

$$\delta S = \int dt \left[\frac{\partial \mathcal{L}}{\partial q} \delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}} (\delta \dot{q}) \right]$$

$$= \int dt \, \delta q(t) \left[\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} \right]$$

+ boundary term

So the solution to the equation of motion extremizes the action.

$$\to \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} = 0$$

Furthermore, to write down a theory, we only need to know the Lagrange density $\mathcal{L}$ and the dynamics follows from extremizing the action.

This is in many cases easier, since the Lagrange density is a scalar under Lorentz transformations.

So far we discussed only a single particle in 1D, but the concept generalizes, e.g. to fields in 3+1 dimensions.

Electrodynamics:

$$\mathcal{L}(A, \partial_\mu A) = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + A_\mu J^\mu$$

and $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, $A(x^\mu)$

EoM :

$$S = \int d^4x \; \mathcal{L}$$

$\delta S = 0$:
$$\frac{\partial \mathcal{L}}{\partial A^\mu} - \frac{d}{dx^\nu} \frac{\partial \mathcal{L}}{\partial(\partial_\nu A^\mu)} = 0$$

$\Rightarrow$ Maxwell's eq.