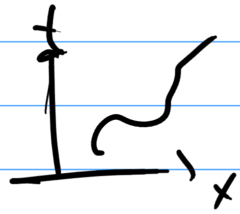
Energy and momentum

Let's say you have a particle moving and the path is given by $\vec{x}(t)$

$$x^\mu = \begin{pmatrix} t(\lambda) \\ \vec{x}(\lambda) \end{pmatrix} \qquad \lambda = t$$



proper time:

$$ds^2 = dx^\mu \eta_{\mu\nu} dx^\nu = -dt^2 + dx^2$$

$$d\tau^2 = -\overset{< 0}{ds^2}$$

$\underline{\lambda = \tau}$:

$$\boxed{\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} = \frac{ds^2}{d\tau^2} = -1}$$

Energy, momentum:

$$\vec{p} \simeq m\dot{\vec{x}}\gamma \; ; \quad \frac{dx^\mu}{d\tau} = \gamma \begin{pmatrix} 1 \\ v \end{pmatrix}$$

$$p^\mu = m\frac{dx^\mu}{d\tau}$$

$$p^0 = \gamma m = m\sqrt{1/_{1-v^2}} \simeq \underbrace{m}_{\text{rest energy}} + \underbrace{\tfrac{1}{2}mv^2}_{\text{kinetic energy}} + \dots$$

Evidently:

$$p_\mu p^\mu = -m^2$$

[onshell condition, dispersion relation] forces:

If we believe that everything is Lorentz covariant, we can start in the restframe of the particle and then boost the result

$$\vec{f} = m\vec{a} = m \frac{d^2\vec{x}}{dt^2}$$

The acceleration has a strange transformation property. It coincides with

$$\frac{d^2\vec{x}}{d\tau^2} \quad \text{when} \quad \frac{d\vec{x}}{d\tau} = 0 \quad (\tau \simeq t)$$

$$\Rightarrow \qquad f^\mu = m \frac{d^2 x^\mu}{d\tau^2}$$

where

$$f^i = \vec{f}$$ in the restframe

$$f^0 = 0$$

Notice that

$$\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} = -1$$

$$\hookrightarrow \quad \left(\frac{d}{d\tau}\frac{dx^\mu}{d\tau}\right)\eta_{\mu\nu}\frac{dx^\nu}{d\tau} = 0$$

$$\rightarrow \quad f^\mu \eta_{\mu\nu} \frac{dx^\nu}{d\tau} = 0$$

$$\Leftrightarrow \quad f^0 = 0 \quad \text{in the restframe}$$

So any consistent force has to fulfill the constraint

$$f^\mu \frac{dx^\nu}{d\tau} \eta_{\mu\nu} = 0$$

Example: particle moving in an electric field:

$$\frac{d\vec{p}}{dt} = q\vec{E} + q\vec{v} \times \vec{B}$$

The generalization reads

$$f^\mu = \frac{dp^\mu}{d\tau} = m \frac{d^2x^\mu}{d\tau^2}$$

$$= q \cdot \eta_{\beta\gamma} F^{\mu\beta} \frac{dx^\gamma}{d\tau}$$

$$\eta_{\mu\nu} f^\mu \frac{dx^\nu}{d\tau} \overset{!}{=} 0$$ [check consistency]

$$= q F^{\alpha\beta} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \eta_{\mu\alpha} \eta_{\nu\beta}$$

$$= 0 \quad (F \text{ antisym.})$$

Currents and densities

The current and density of a set of moving particles is given by

$$\rho(\vec{x},t) = \sum_n \overset{\text{charges}}{e_n}\, \delta^3(\vec{x}-\vec{x}_n(t))$$

$$\vec{j}(\vec{x},t) = \sum_n e_n\, \delta^3(\vec{x}-\vec{x}_n(t))\, \frac{d\vec{x}_n}{dt}$$

This can be cast into a four-vector

$$j^\mu = \sum_n e_n\, \delta^3(\vec{x}-\vec{x}_n)\, \frac{dx_n^\mu(t)}{dt}$$

In order to show that this is indeed a four-vector, we write

$$j^\mu(\vec{x},t) = \sum_n \int dt'\, e_n\, \delta^4(x^\mu - x_n^\mu(t))\, \frac{dx_n^\mu}{dt}$$

$$x_n^0 = t'$$

$$j^\mu(\vec{x},t) = \sum_n \int d\lambda\, e_n\, \delta^4(x^\mu - x_n^\mu(\lambda))\, \frac{dx_n^\mu}{d\lambda}$$

Also notice (see exercises)

$$\partial_\mu j^\mu = 0$$