Construction of maximally symmetric spaces

We have seen that maximally symmetric spaces are unique as long as they agree in signature and curvature.

Hence it is sufficient to construct a prototype for these spaces. All other maximally symmetric spaces will be related by coordinate transformations.

Consider a maximally symmetric space with signature

$$\eta_{\mu\nu} = \begin{pmatrix} \pm & & & \\ & \pm & & \\ & & \ddots & \\ & & & \pm \end{pmatrix}$$

and the following embedding

$$z^2 + x^\mu x^\nu \eta_{\mu\nu} = r^2$$

into the space with N+1 dimensions and line element

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu + dz^2$$

The constraint turns into

$$z\, dz + x^\mu dx^\nu \eta_{\mu\nu} = 0$$

and hence

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu + \frac{\eta_{\mu\nu}\, dx^\mu x^\nu\, \eta_{\alpha\beta}\, dx^\alpha x^\beta}{r^2 - \eta_{\mu\nu} x^\mu x^\nu}$$

The induced metric is accordingly

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{x^\alpha \eta_{\mu\alpha}\ x^\beta \eta_{\nu\beta}}{r^2 - \eta_{\mu\nu} x^\mu x^\nu}$$

This metric leads to the Christoffel symbol and Ricci scalar

$$K = \frac{1}{r^2} \quad ; \quad \Gamma^\mu_{\alpha\beta} = K x^\mu g_{\alpha\beta}$$

$$R = N(N-1)K$$

$$R_{\mu\nu\lambda\kappa} = K\left(g_{\mu\lambda} g_{\nu\kappa} - g_{\mu\kappa} g_{\nu\lambda}\right)$$

This is a maximally symmetric space.

The symmetries are inherited from the N+1 dimensional space:

$$\bar{g}_{MN} = \begin{pmatrix} \eta_{\mu\nu} & \\ & 1 \end{pmatrix} \qquad x_M = \begin{pmatrix} x_\mu \\ z \end{pmatrix}$$

$$\bar{g}'_{MN} = \bar{\Lambda}_M{}^L \bar{\Lambda}_N{}^K \bar{g}_{LK}$$

A) Transformations that leave z invariant lead to rotations/Lorentz-transformations in N dimensions

$$\bar{\Lambda}_M{}^N = \begin{pmatrix} \Lambda_\mu{}^\nu & \\ & 1 \end{pmatrix}$$

$$x'_\mu = \Lambda_\mu{}^\nu x_\nu$$

$$\Lambda_\mu{}^\lambda \Lambda_\nu{}^\kappa \eta_{\lambda\kappa} = \eta_{\lambda\kappa}$$

B) Quasitranslations

$$x^{\mu\prime} = x^\mu + a^\mu \left[\left(1 - K \eta_{\rho\sigma} x^\rho x^\sigma\right)^{1/2} - bK\eta_{\rho\sigma} x^\rho a^\sigma \right]$$

with

$$b = \frac{1 - \sqrt{1 - K\eta_{\rho\sigma} a^\rho a^\sigma}}{K \eta_{\sigma\rho} a^\rho a^\sigma}$$

That these are really the symmetries has to be checked by calculating the corresponding Killing vectors.

A)

$$x^{\mu\prime} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

with $\Lambda_{\mu}{}^{\nu} \Lambda_{\kappa}{}^{\lambda} \eta_{\nu\lambda} = \eta_{\mu\kappa}$

$\hookrightarrow$ $\Lambda_{\mu}{}^{\nu} \sim \delta_{\mu}{}^{\nu} + \Omega_{\mu}{}^{\nu}$

with $\Omega_{\mu}{}^{\nu} \eta_{\nu\kappa} + \Omega_{\kappa}{}^{\nu} \eta_{\mu\nu} = 0$

The Killing vector is then

$$x^{\mu\prime} = x^{\mu} + \xi^{\mu}$$

$$\xi^{\mu} = \Omega^{\mu}{}_{\nu} x^{\nu}$$

$$\xi_{\mu} = g_{\mu\nu} \Omega^{\nu}{}_{\kappa} x^{\kappa}$$

We find then

$$D_{\nu} \xi_{\mu} = g_{\mu\lambda} \Omega^{\lambda}{}_{\nu} + \Gamma^{\lambda}_{\mu\nu} \xi_{\lambda} + \partial_{\nu} g_{\mu\lambda} \Omega^{\lambda}{}_{\kappa} x^{\kappa}$$

Which turns out to be antisymmetric in $\nu \leftrightarrow \mu$

after some algebra.

B) Likewise, for the quasitranslations

$$x^{\mu'} = x^{\mu} + a^{\mu} \left(\sqrt{1 - k x^{\rho} x^{\sigma} \eta_{\rho\sigma}} + O(a) \right)$$

$$\hookrightarrow \quad \xi^{\mu} = a^{\mu} \sqrt{1 - k x^{\rho} x_{\rho}}$$

$$\xi_{\nu} = g_{\nu\mu} \xi^{\mu}$$

and

$$D_{\nu} \xi_{\mu} + D_{\mu} \xi_{\nu} = 0$$

Can be checked explicitly.

Geodesic equation:

$$\frac{\partial^2 x^\mu}{\partial \tau^2} + \Gamma^\mu{}_{\alpha\beta} \frac{\partial x^\alpha}{\partial \tau} \frac{\partial x^\beta}{\partial \tau} = 0$$

Using the expression for the Christoffel symbol

$$\Gamma^\mu{}_{\alpha\beta} = k\, x^\mu g_{\alpha\beta}$$

and the normalization of the geodesic

$$\frac{\partial x^\alpha}{\partial \tau} \frac{\partial x^\beta}{\partial \tau} g_{\alpha\beta} = 1$$

this simplifies to

$$\frac{\partial^2 x^\mu}{\partial \tau^2} + k\, x^\mu = 0$$

Solutions of this equation are given by

$$x^\mu = \frac{1}{\sqrt{k}} e^\mu \sin \sqrt{k}\, \tau$$

$$= r\, e^\mu \sin \tau/r$$

where the initial conditions are given by

$$\left.\frac{dx^\mu}{d\tau}\right|_{\tau=0} = e^\mu \,, \quad e^\mu e^\nu g_{\mu\nu} = 1$$

Above construction obviously leads only to positive curvature

$$K = \frac{1}{r^2}$$

To obtain a space with negative curvature, one can follow the same steps with the constraint

$$r^2 = z^2 - x^\mu x^\nu \eta_{\mu\nu}$$

and the N+1 dimensional line element

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu - dz^2$$

which leads to the embedded metric

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{\eta_{\mu\alpha} x^\alpha \eta_{\nu\beta} x^\beta}{r^2 + \eta_{\alpha\beta} x^\alpha x^\beta}$$

with curvature $K = -1/r^2$

The flat case is obtained from either metric in the limit

$$r \to \infty$$

which gives

$$g_{\mu\nu} = \eta_{\mu\nu}$$

Notice that this metric can also be brought to the more familiar form from cosmology by using the coordinate transformation (K>0)

$$t = \frac{1}{\sqrt{K}} \left[\frac{Kx}{2} \cosh(\sqrt{K} t') + \left(1 + \frac{Kx^2}{2}\right) \sinh(\sqrt{K} t) \right]$$

$$x^2 = \vec{x} \cdot \vec{x} \qquad ; \; \eta_{\mu\nu} = \begin{pmatrix} 1 & \\ & -\mathbb{1} \end{pmatrix}$$

$$x^\mu = x'^\mu \cdot \exp(\sqrt{K} t)$$

In the new coordinates the line elements reads

$$ds^2 = dt'^2 - a^2(t')\, dx' \cdot dx'$$

with

$$a(t) = \exp \sqrt{K} t$$

Reminder: dS space in cosmology

Remember that we found this metric earlier as a solution to the Einstein equation with a cosmological constant.

Lagrangian/action

$$S = M_{Pl}^2 \int d^4x \sqrt{g} \left\{ R + \lambda \right\}$$

$$\hookrightarrow \quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = T_{\mu\nu} \times 8\pi G_N$$

In vacuum (CC only)

$$R = 4\lambda = N(N-1) K = 12K$$

$$\rightarrow \quad \lambda = 3K$$

$\lambda > 0$ de Sitter

$\lambda < 0$ anti-de Sitter

$\lambda = 0$ Minkowski