# Constraints on maximally symmetric spaces

Any vector/tensor fulfills

$$(\nabla_\rho\nabla_\sigma - \nabla_\sigma\nabla_\rho)\, V_\mu = -R^\lambda{}_{\mu\sigma\rho}\, V_\lambda$$

$$(\nabla_\rho\nabla_\sigma - \nabla_\sigma\nabla_\rho)\, T_{\mu\nu} = -R^\lambda{}_{\mu\sigma\rho}\, T_{\lambda\nu} - R^\lambda{}_{\nu\sigma\rho}\, T_{\mu\lambda}$$

This is also true for the second derivative of the Killing vectors

$$(\nabla_\nu\nabla_\sigma - \nabla_\sigma\nabla_\nu)\, \nabla_\mu\xi_\rho = -R^\lambda{}_{\rho\sigma\nu}\, \nabla_\mu\xi_\lambda - R^\lambda{}_{\mu\sigma\nu}\, \nabla_\lambda\xi_\rho$$

We have seen before, that

$$\nabla_\sigma\nabla_\rho\, \xi_\mu = -R^\lambda{}_{\sigma\rho\mu}\, \xi_\lambda$$

Taking the covariant derivative of this relation then can be used to relate the covariant derivative of the Riemann tensor to itself.

$$R^{\lambda}{}_{\sigma\rho\mu} D_\nu \xi_\lambda - R^{\lambda}{}_{\nu\rho\mu} D_\sigma \xi_\lambda$$

$$+ \left( D_\nu R^{\lambda}{}_{\sigma\rho\mu} - D_\sigma R^{\lambda}{}_{\nu\rho\mu} \right) \xi_\lambda$$

$$= - R^{\lambda}{}_{\rho\sigma\nu} \cdot D_\mu \xi_\lambda - R^{\lambda}{}_{\mu\sigma\nu} D_\lambda \xi_\rho$$

Using the Killing equation and grouping the different terms gives:

$$\left[ - R^{\lambda}{}_{\rho\sigma\nu} \delta^{\kappa}_{\mu} + R^{\lambda}{}_{\mu\sigma\nu} \delta^{\kappa}_{\rho} - R^{\lambda}{}_{\sigma\rho\mu} \delta^{\kappa}_{\nu} + R^{\lambda}{}_{\nu\rho\mu} \delta^{\kappa}_{\sigma} \right] D_\kappa \xi_\lambda$$

$$= \left[ D_\nu R^{\lambda}{}_{\sigma\rho\mu} - D_\sigma R^{\lambda}{}_{\nu\rho\mu} \right] \xi_\lambda$$

This is true for every Killing vector of the metric.

But if the space is maximally symmetric, there is an isometry about every point. This means that the matix in front of

$$D_\kappa \xi_\lambda$$

vanishes in every point.

Taking the contraction with $\delta^{\mu}{}_{\lambda}$ yields

$$-N R^{\lambda}{}_{\rho\sigma\nu} + R^{\lambda}{}_{\rho\sigma\nu} - R^{\lambda}{}_{\sigma\rho\nu} + R^{\lambda}{}_{\nu\rho\sigma}$$
$$= - R^{\lambda}{}_{\rho\sigma\nu} + R_{\sigma\rho}\,\delta^{\lambda}{}_{\nu} - R_{\nu\rho}\,\delta^{\lambda}{}_{\sigma}$$

Using the cyclicity of R and the symmetries, one finds

$$\boxed{(N-1)\, R_{\lambda\rho\sigma\nu} = R_{\nu\rho}\, g_{\lambda\sigma} - R_{\sigma\rho}\, g_{\lambda\nu}}$$

Contracting once more gives

$$\boxed{R_{\sigma\rho} = \frac{1}{N}\, g_{\sigma\rho}\, R^{\lambda}{}_{\lambda}}$$

and finally

$$R_{\lambda\rho\sigma\nu} = \frac{R^{\lambda}{}_{\lambda}}{N(N-1)}$$
$$\times \left\{ g_{\nu\rho}\, g_{\lambda\sigma} - g_{\sigma\rho}\, g_{\lambda\nu} \right\}$$

In principle, the Ricci scalar R could still be space-dependent. But using the Bianchi identity one finds:

$$\nabla_\eta R_{\lambda\mu\nu\kappa} + \nabla_\kappa R_{\lambda\mu\eta\nu} + \nabla_\nu R_{\lambda\mu\kappa\eta} = 0$$

$$\hookrightarrow \nabla_\lambda \left( R^\lambda{}_\eta - \frac{1}{2} \delta^\lambda{}_\eta R \right) = 0$$

Hence for maximally symmetric spaces:

$$\left( \frac{2}{N} - 1 \right) \nabla_\sigma R^\lambda{}_\lambda = 0$$

And hence R=const for N>2.

# Uniqueness of maximally symmetric spaces

We have seen that maximally symmetric spaces have a very simple form for the Riemann tensor, namely

$$R_{\mu\nu\lambda\kappa} = K\left(g_{\mu\lambda}g_{\nu\kappa} - g_{\mu\kappa}g_{\nu\lambda}\right)$$

Now assume that you have two maximally symmetric spaces with the same signature and the same curvature K.

In this case, the two spaces are actually locally equivalent, so they are related by a coordinate transformation.

$$g'_{\mu\nu} = \frac{\partial x^\beta \partial x^\alpha}{\partial x'^\nu \partial x'^\mu} g_{\alpha\beta}$$

This can be shown by constructing the coordinate transformation recursively/by induction.

To lowest order we mimick the proof of a free falling frame.

We can also just consider transformations that take the origin in the first space to the origin in the second space, due to homogeneity.

So in leading order

$$x'^{\mu} = d^{\mu}{}_{\nu} x^{\nu}$$

And we require

$$g'_{\mu\nu}(x')\Big|_{x'=0} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}(x)\Big|_{x=0}$$

$$\hookrightarrow \quad g'_{\mu\nu}(0) = d^{\alpha}{}_{\mu} d^{\beta}{}_{\nu} g_{\alpha\beta}(0)$$

Again, this is always possible as long as the signatures are the same and this will give

$d^{\mu}{}_{\nu}$.

Likewise, we have seen that in the free falling frame the first derivative of the metric vanishes.

So connecting both manifolds to the free falling frame, we can construct a coordinate transformation that relates the first derivatives of the metrices.

Explicitly this reads $x'^\mu = d^\mu_\nu x'^\nu + \frac{1}{2} \bar{d}^\mu_{\nu\lambda} x'^\nu x'^\lambda$

$$g'_{\mu\nu}(0) + x'^\lambda \partial_\lambda g'_{\mu\nu}$$

$$\overset{!}{=} \left(d_\mu^\alpha + \bar{d}_{\mu\lambda}^\alpha x^\lambda\right)\left(d_\nu^\beta + \bar{d}^\beta_{\nu\lambda} x^\lambda\right)$$

$$\times \left(g_{\alpha\beta} + x^\lambda \partial_\lambda g_{\alpha\beta}\right)$$

So in next to leading order

$$x'^\lambda \partial_\lambda g'_{\mu\nu} = x'^\lambda \left(d_\mu^\alpha \bar{d}^\beta_{\lambda\nu} + d_\nu^\alpha \bar{d}^\beta_{\lambda\mu}\right) g_{\alpha\beta}$$

$$+ x^\lambda \partial_\lambda g_{\alpha\beta}$$

which can always be solved for $\bar{d}^\alpha_{\mu\nu}$

[check the old lecture if this point is unclear]

Higher orders can obviously not always be related, since this can only work if the two spaces are really related by a coordinate transformation.

Still, we can try. Consider the transformation rule of the Christophel symbol.

$$\frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\lambda} = \frac{\partial x'^\mu}{\partial x^\kappa} \Gamma^\kappa_{\lambda\rho}(x) - \frac{\partial x'^\nu}{\partial x^\rho} \frac{\partial x'^\kappa}{\partial x^\lambda} \Gamma'^\mu_{\nu\kappa}(x')$$

Apparently, this relationship could be used to solve for the higher order terms in the expansion x'(x) in terms of the expansion of the metrices and lower ones. Notice that the right side only contains first derivatives dx'/dx.

However, this relation only can solved if the right side is a derivative. To test this, we can take another derivative and see if the outcome is symmetric, i.e.

$$\frac{\partial}{\partial x^\sigma} \left[ \frac{\partial x'^\mu}{\partial x^\kappa} \Gamma^\kappa_{\lambda\rho} - \frac{\partial x'^\nu}{\partial x^\rho} \frac{\partial x'^\kappa}{\partial x^\lambda} \Gamma'^\mu_{\nu\kappa}(x') \right]$$

$$= \sigma \leftrightarrow \lambda$$

Shuffeling the terms around, this reads

$$\frac{\partial x'^{\mu}}{\partial x^{\kappa}} R^{\kappa}{}_{\rho\lambda\eta} = \frac{\partial x'^{\nu}}{\partial x_{\rho}} \frac{\partial x'^{\kappa}}{\partial x^{\lambda}} \frac{\partial x'^{\sigma}}{\partial x^{\eta}} R'^{\mu}{}_{\nu\kappa\sigma}$$

This seems like a tautology since in
order to show that we can construct higher
and higher orders, we seem to need the
information that the two Riemann tensors
are related by a coordinate transformation.

This seems like dead end, since checking this
will require second order derivative terms

$$\frac{\partial^2 x^{\mu}}{\partial x \partial x} \in R'\left(g' = \frac{\partial x}{\partial x'} \frac{\partial x}{\partial x'} g\right)$$

Here, the relation from the last section comes
to the rescue. We know for both spaces that

$$R_{\mu\nu\kappa\lambda} = k\left(g_{\mu\kappa} g_{\nu\lambda} - g_{\mu\lambda} g_{\nu\kappa}\right)$$

$$R' = k\left( \qquad \right)$$

So the constraint turns into

$$\frac{\partial x'^{\mu}}{\partial x^{\eta}} g_{\lambda\rho} - \frac{\partial x'^{\mu}}{\partial x^{\lambda}} g_{\rho\eta}$$

$$= \frac{\partial x'^{\nu}}{\partial x^{\rho}} \left( \frac{\partial x'^{\kappa}}{\partial x^{\lambda}} \frac{\partial x'^{\mu}}{\partial x^{\eta}} g'_{\nu\kappa} - \frac{\partial x'^{\mu} \partial x'^{\sigma}}{\partial x^{\lambda} \partial x^{\eta}} g'_{\nu\sigma} \right)$$

This can be recursively checked, since this expression only contains terms of lower order in the expansion

$$x'^{\mu} = a^{\mu}_{\alpha} x^{\alpha} + \frac{1}{2} a^{\mu}_{\alpha\beta} x^{\alpha} x^{\beta} + a^{\mu}_{\alpha\beta\gamma} \frac{x^{\alpha} x^{\beta} x^{\gamma}}{6} + \ldots$$

This concludes the proof.
So any two maximally symmetric spaces with the same signatures and curvature are related by coordinate transformations.

In particular, this means that if we can construct a maximally symmetric space with arbitrary signature and curvature, this represents all maximally symmetric spaces.