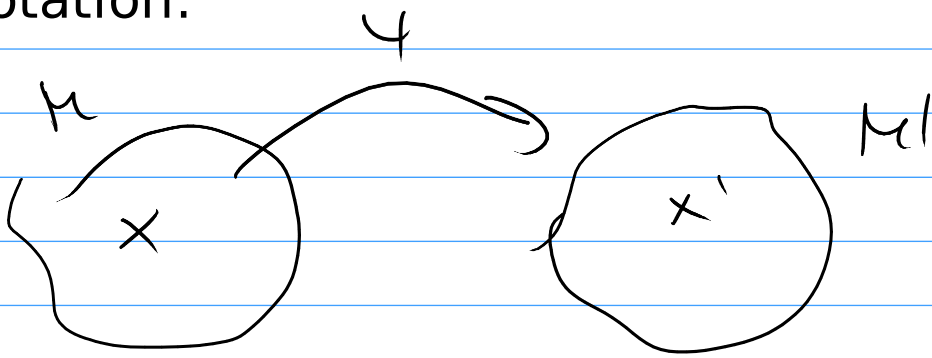
Killing vectors

We want to discuss symmetries of a metric. The question is: how can we do that without introducing a coordinate system first. Is there a coordinate independent (=covariant) way of doing it?

notation:



$$x' = \psi(x)$$

Transformation properties of a scalar:

$$\varphi(x) \mapsto \varphi'(x') = \varphi(x)$$

this is common but sloppy notation.
What we really mean is

$$\varphi'(x') = \varphi(\psi^{-1}(x'))$$

$$= \varphi \circ \psi^{-1}(x')$$

Likewise, the metric transforms as

$$g'_{\mu\nu}(x') = \Lambda^{\lambda}{}_{\mu}\Lambda^{\kappa}{}_{\nu}\, g_{\lambda\kappa} \circ \varphi^{-1}(x')$$

where $\Lambda^{\lambda}{}_{\mu} = \frac{\partial x^{\lambda}}{\partial x'^{\mu}}(x')$

We identify symmetries in the metric by requiring that it is forminvariant.
This means it has the same form in the new coord. system with the new coordinates as in the old one.

$$g'_{\mu\nu}(x') \overset{!}{=} g_{\mu\nu}(x')$$

We don't want to impose that the metric transforms as a scalar, which would mean

$$g'_{\mu\nu}(x') = g_{\mu\nu} \circ \varphi^{-1}(x')$$

As an example consider the line element

$$ds^2 = dx^i dx^j \eta_{ij} + (dx^i x^j \eta_{ij})^2 A$$

with the metric

$$g_{ij} = \eta_{ij} + A x_i x_j$$

When we perform a rotation $x' = O \cdot x$

$$x'_i = O_{ij} x_j$$

The metric transforms as

$$g'_{ij}(x') = O_{ik} O_{jk} \left(\eta_{ij} + A (O^T x')_i (O^T x')_j \right)$$
$$= \eta_{ij} + A x'_i x'_j$$

So g' has the same form as a function of x' as g as a function of x.

This does not mean that g is scalar. This would read

$$g'(x') \overset{?}{=} g \circ \varphi^{-1}(x')$$
$$= \eta_{ij} + A (O^T x')_i (O^T x')_j$$

So for a forminvariant metric, the transformation properties of the Lorentz indices just compensates the mapping x->x'.

Let's now follow theses steps for an infinitesimal transformation

$$x'^{\mu} = x^{\mu} + \epsilon \xi^{\mu}$$

Forminvariance then reads

$$0 = \frac{\partial \xi^{\mu}}{\partial x^{\rho}} g_{\mu\nu} + \frac{\partial \xi^{\nu}}{\partial x^{\sigma}} g_{\rho\nu} + \xi^{\mu} \frac{\partial g_{\rho\sigma}}{\partial x^{\mu}}$$

This can be more compactly written for the covariant vector $\xi_{\nu} = g_{\nu\mu}\xi^{\mu}$

$$0 = \frac{\partial \xi_{\sigma}}{\partial x^{\rho}} + \frac{\partial \xi_{\rho}}{\partial x^{\sigma}} + \xi^{\mu}\left(\frac{\partial g_{\rho\sigma}}{\partial x^{\mu}} - \frac{\partial g_{\mu\sigma}}{\partial x^{\rho}} - \frac{\partial g_{\rho\mu}}{\partial x^{\sigma}} \right)$$

$$= \frac{\partial \xi_{\sigma}}{\partial x^{\rho}} + \frac{\partial \xi_{\rho}}{\partial x^{\sigma}} - 2\xi_{\mu}\Gamma^{\mu}{}_{\rho\sigma}$$

$$\boxed{= \nabla_{\rho}\xi_{\sigma} + \nabla_{\sigma}\xi_{\rho} = 0}$$

This is the Killing equation

Obviously this is a covariant equation. So this gives us the possibility to find symmetries of metrices in a coordinate independent way. We just have to check the existence of Killing vectors.

Killing vectors are uniquely determined by their first and second derivatives

If $\xi_\mu(x)$ and $D_\nu \xi_\mu(x)$ is known

in a point X, one can reconstruct it in a neighborhood of X.

This follows from:

A) $[D_\rho, D_\sigma]\xi_\mu = -R^\lambda{}_{\mu\rho\sigma}\xi_\lambda$

since ξ is a vector

B) R is cyclic

c) the Killing equation

Altogether:

$$[D_\rho, D_\sigma]\xi_\mu = -[D_\mu, D_\rho]\xi_\sigma - [D_\sigma, D_\mu]\xi_\sigma$$

$$\hookrightarrow [D_\rho, D_\sigma]\xi_\mu = D_\mu D_\sigma \xi_\sigma = -R^\lambda{}_{\rho\sigma\mu}\xi_\lambda$$

This means the second derivative
can be written as a function of the Riemann
tensor and the 0th/1st derivatives.

Besides the Killing equation is linear.

Hence there the Killing vector is uniquely
determined by the 0th and 1st derivative
in any point.

-> there are at most

$$\underset{\xi_\mu|_x}{N} + \underset{D_\mu \xi_\nu|_x}{\frac{N(N-1)}{2}} = \underset{(\text{antisym})}{\frac{N(N+1)}{2}}$$

Examples:

3D Euclidean space: 3 rot, 3 translations

4D Minkowski: 4 trans, 3 boost, 3 rotations

4D Euclidean: 4 trans, 6 rotations

Homogeniety and Isotropy

A space is homogeneous iff there is an isometry that can carry every point to arbitrary points in the neighborhood. These are called translations

One handy basis for translations are

$$\xi^{(\mu)}_{\lambda}\Big|_X = \delta_\lambda^{\mu}$$

$$D_\nu \xi^{(\mu)}_{\lambda}\Big|_X = 0$$

A space is isotropic about a point X iff there is a isometry that takes X to X but the derivatives are arbitrary.

$$\xi^{(\mu\nu)}_{\lambda}\Big|_X = 0$$

$$D_\rho \xi^{(\mu\nu)}_{\lambda}\Big|_X = \delta_\lambda^{\mu}\delta_\rho^{\nu} - \delta_\rho^{\mu}\delta_\lambda^{\nu}$$

If a space is isotropic about every point
it is automatically homogeneous.

This is because the different isotropies
can be used to construct a tranlation

$$x^\mu \to x^{\mu'} = x^\mu + \xi^\mu$$
$$\simeq x^\mu + (x^\nu - X^\nu) D_\nu \xi^\mu$$

So the difference between a isotropy around
X=0 and X=a is

$$a^\nu D_\nu \xi^\mu = a^\nu D_\nu \xi^{\mu}(x,X)$$
$$= a^\nu \left(\delta_\nu^\lambda \eta^{\mu\kappa} - \delta_\nu^\kappa \eta^{\mu\lambda} \right)$$

These can be used to construct arbitrary
translations.

$$x^\mu \to x^{\mu'} + \left(\delta^\mu_\lambda \delta^\nu_\kappa - \delta^\mu_\kappa \delta^\nu_\lambda \right) a_\nu b^\kappa c^\lambda$$

Hence there are the following equivalences:

maximally symmetric
<-> isotropic and homogeneous
<-> isotropic about every point
<-> N(N+1)/2 Killing vectors

For flat space, the covariant derivatives become normal derivatives and the Killing vectors simply read

$$\xi^{(\nu)}_{\mu} = \delta^{\nu}_{\mu}$$

$$\xi^{(\nu\mu)}_{\lambda} = \delta^{\mu}_{\lambda}\,(x^{\nu} - X^{\nu}) - \delta^{\nu}_{\lambda}\,(x^{\mu} - X^{\mu})$$

for arbitrary X.

This follows from $\partial_{\rho}\partial_{\nu}\,\xi = 0$

and the boundary conditions discussed before.