How does the Einstein equation look like when formulated using the Vielbein?

Consider the functional derivative of the Lagrange density:

$$\frac{\partial \mathcal{L}}{\partial V^\mu{}_\alpha} \qquad : \qquad \frac{\delta S}{\delta V^\mu{}_\alpha} = 0$$

Consider the parts that can be written in terms of the metric, like the Einstein-Hilbert term:

$$M_{pe}^2 \int d^4x \sqrt{g}\, R$$

Then:

$$\frac{\delta \mathcal{L}}{\delta V^\mu{}_\alpha} = \frac{\partial \mathcal{L}}{\partial g^{\lambda\kappa}} \frac{\partial g^{\lambda\kappa}}{\partial V^\mu{}_\alpha}$$

$$\frac{\partial g^{\lambda\kappa}}{\partial V^\mu{}_\alpha} = \frac{\delta}{\delta V^\mu{}_\alpha} \left(V^\lambda{}_\beta V_\gamma{}^\kappa \eta^{\beta\gamma} \right)$$

$$= \delta^\lambda{}_\mu \delta^\alpha{}_\beta V_\gamma{}^\kappa \eta^{\beta\gamma}$$

$$+ V^\lambda_\beta \delta_\mu{}^\kappa \delta^\gamma{}_\alpha \eta^{\beta\gamma}$$

And hence:

$$\frac{\partial g^{\lambda\kappa}}{\partial V^{\mu}{}_{\alpha}} V_{\nu\alpha} = \delta^{\lambda}{}_{\mu}\,\delta^{\kappa}{}_{\nu} + \delta^{\lambda}{}_{\nu}\,\delta^{\kappa}{}_{\mu}$$

$$\frac{\partial \mathcal{L}}{\partial V^{\mu}{}_{a}} V_{\nu a} = 2\,\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}}$$

This enters in the energy-momentum tensor of GR

This will enter the generalization of the energy-momentum tensor when the vielbein is used

Word auf caution: It is not obvious that this construction leads to a symmetric energy-momentum tensor. But one can prove that its does.

How to construct the covariant derivative for the vielbeins

We would like to construct a covariant derivative, such that

$$D_\beta A_\alpha$$

transforms like a tensor

$$D_\beta A_\alpha \to \Lambda_\beta{}^\gamma \Lambda(x)_\alpha{}^\delta D_\gamma A_\delta$$

$$A_\alpha \to \Lambda(x)^\beta{}_\alpha A_\beta$$

The construction follows the same steps as in GR.

We can do the same approach for general representations, so consider:

$$\psi \to D(\Lambda(x))\psi$$

We would like to construct a covariant derivative such that

$$\nabla_\alpha \psi$$

transforms as

$$\nabla_\alpha \psi_n \to D(\Lambda)_{nm} \Lambda^\beta{}_\alpha (D_\beta \psi_m)$$

$$\psi_n \to D(\Lambda^{(x)})_{nm} \psi_m$$

Ultimately, one finds for the covariant derivatives:

$$D_\alpha = V_\alpha{}^\mu \left(\partial_\mu \mathbb{1}_{mn} + \frac{1}{2} \sigma^{\beta\gamma}_{mn} V_\beta{}^\nu \left(\frac{\partial}{\partial x^\mu} V_{\gamma\nu}\right)\right)$$

Using this we finally write down the Lagrangian for a spinor

$$\mathcal{L} \supset \bar{\psi}(\gamma^\alpha D_\alpha - m)\psi$$

$\hookrightarrow$ This can now be used to determine:
- EoM of the spinor
- a contribution to the energy-momentum in the Einstein equation

Penrose diagrams:

Penrose diagrams are digrams that display the causal structure of a space. They make it easy to track the paths followed by light.

For example: Minkowski space:

$$ds^2 = -dt^2 + dx^2 \quad (1D)$$

$$= -du\,dv \qquad u = t-x, \quad v = t+x, \quad (u,v) \in (-\infty, \infty)$$

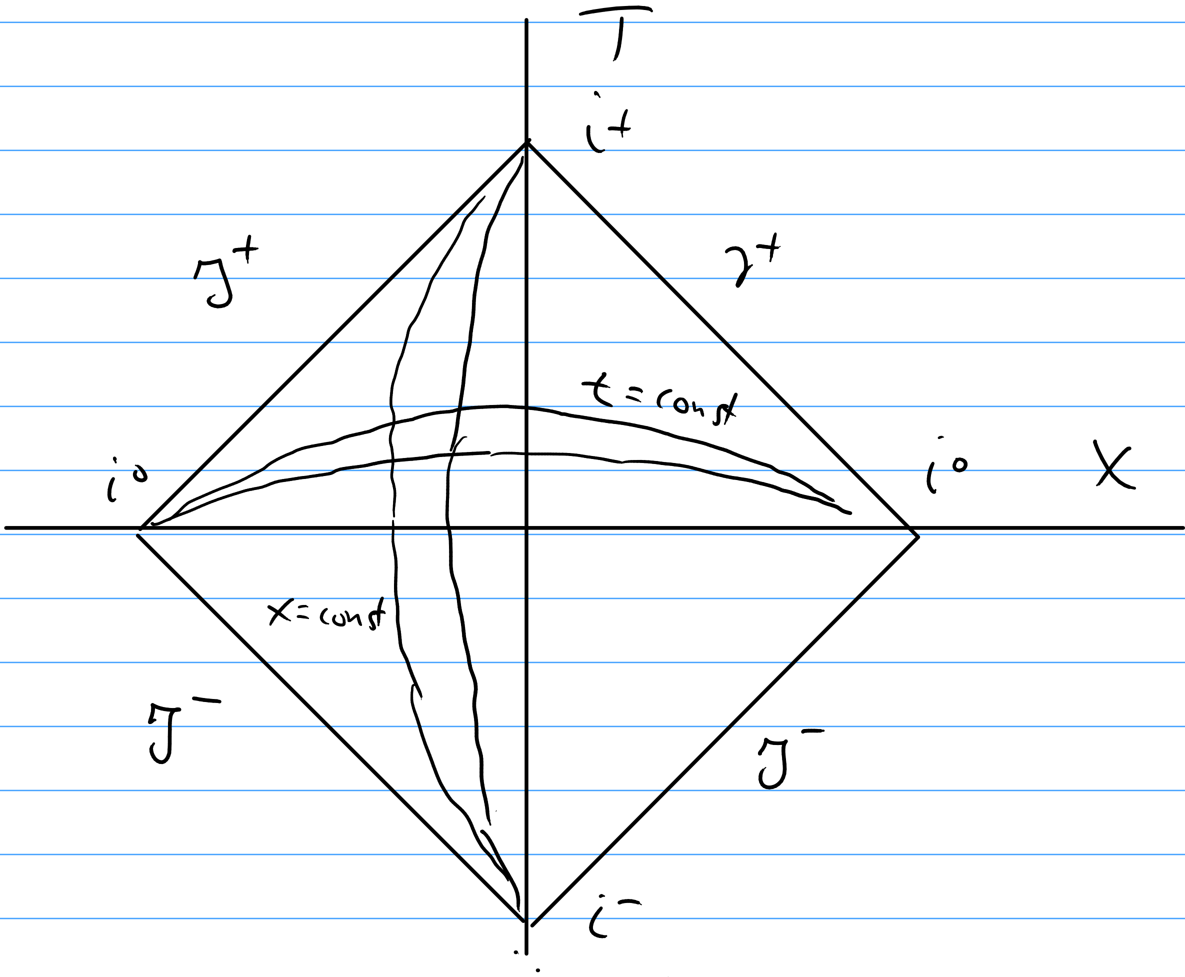
$p = \arctan v$
$q = \arctan u$
$(p,q) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$= -\frac{dp\,dq}{\cos^2 p \cos^2 q}$$

$T = p+q$
$X = p-q$
$T+X \in [-\pi, \pi]$
$T-X \in [-\pi, \pi]$

$$= \frac{-dT^2 + dX^2}{\cos^2\left(\frac{T+X}{2}\right)\cos^2\left(\frac{T-X}{2}\right)}$$

$$= \Omega(X,T)\left(-dT^2 + dX^2\right)$$



$$ds^2 = 0$$

Light follows: $dT^2 = dX^2$

↳ diagonal lines

i^0	space-like infinity
i^+	time-like + infinity
i^-	time-like - infinity
$\mathscr{J}^+$	light rays end here
$\mathscr{J}^-$	light rays originate here

Another example: deSitter space
(cosmology with a CC)

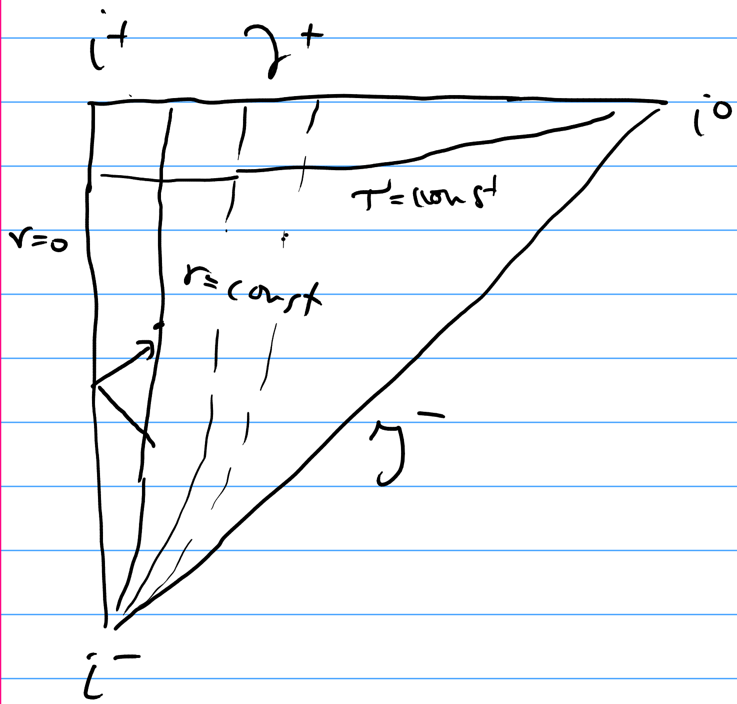
$$a(t) = e^{Ht} \quad ; \quad H = \text{const.}$$

The metric in 3D looks like

$$ds^2 = -dt^2 + a(t)^2\left(dr^2 + \underline{r^2 d\Omega^2}\right)$$

let's neglect.

$$= a(t)^2\left(-d\tau^2 + dr^2\right)$$



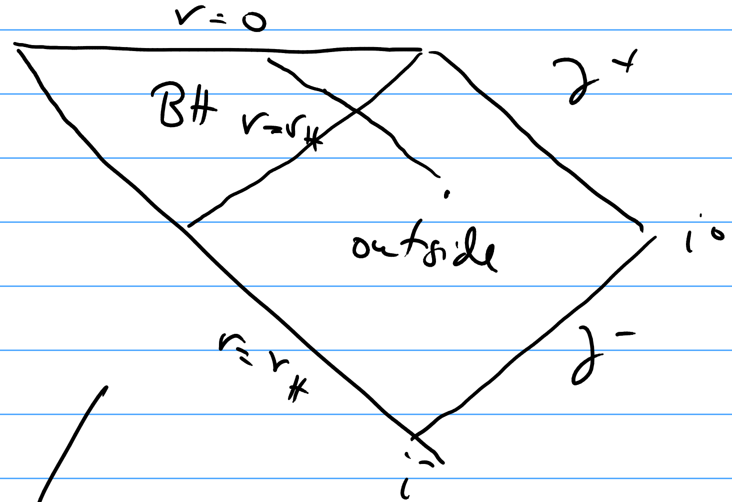
conformal time:

$$a(t)\,d\tau = dt$$
$$d\tau = e^{-Ht}\,dt$$
$$\tau = -\frac{1}{a \cdot H}$$

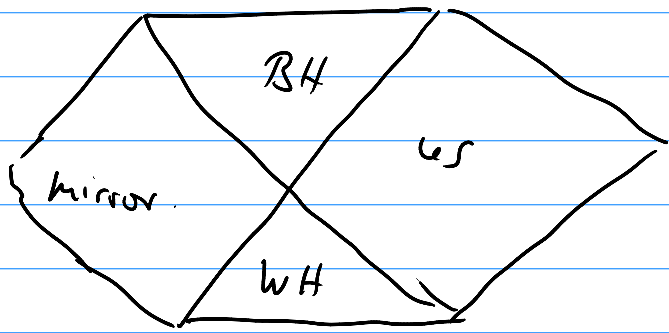
$$a \in [0, \infty]$$
$$\tau \in [-\infty, 0]$$
$$r \in [0, \infty]$$

Notice that there are horizons.

black holes:



complete into



collapsing star: