Irreducible representations of the rotation group

We are interested in the representations of the rotation group with the algebra

$$[J_i, J_j] = i\epsilon_{ijk} J_k \qquad i,j,k \in 1,2,3$$

Since the operators do not commute, we can diagonalize at most one of the them.

However, consider

$$J^2 = \sum_i J_i^2$$

Then

$$[J^2, J_i] = \sum_k [J_k^2, J_i]$$

$$= \sum_k J_k [J_k, J_i] + [J_k, J_i] J_k$$

$$= \sum_k i\epsilon_{ijk} \{ J_k J_j + J_j J_k \}$$

$$= 0$$

Hence we can diagonalize

$$\partial^2 \quad \text{and} \quad \partial_z$$

$$\partial^2 |\alpha, \beta\rangle = \alpha |\alpha, \beta\rangle$$

$$\partial_z |\alpha, \beta\rangle = \beta |\alpha, \beta\rangle$$

Now consider the ladder operators

$$\partial_+ = \partial_x + i\partial_y$$

$$\partial_- = \partial_x - i\partial_y$$

Then

$$[\partial_z, \partial_+] = [\partial_z, \partial_x] + i[\partial_z, \partial_y]$$

$$= i\partial_y + \partial_x = \partial_+$$

also: $[\partial_z, \partial_-] = -\partial_-$

and

$$[\partial_+, \partial_-] = 2\partial_z$$

and obviously $[\partial_\pm, \partial^2] = 0$

What are the quantum numbers of

$$\partial_+ |\alpha,\beta\rangle \quad \text{and} \quad \partial_- |\alpha,\beta\rangle \ ?$$

$$\partial^2 \partial_\pm |\alpha,\beta\rangle = \partial_\pm \partial^2 |\alpha,\beta\rangle$$
$$= \alpha\, \partial_\pm |\alpha,\beta\rangle$$

But

$$\partial_z \partial_+ |\alpha,\beta\rangle = \partial_+ + \partial_+ \partial_z |\alpha,\beta\rangle$$
$$= (\beta+1)\, \partial_+ |\alpha,\beta\rangle$$

Hence

$$\partial_+ |\alpha,\beta\rangle = N\, |\alpha,\beta+1\rangle$$

Likeswise

$$\partial_+ |\alpha,\beta\rangle = N\, |\alpha,\beta-1)$$

Could N=0 at some point and terminate the series?

Now consider the norm of $\partial_+ |\alpha, \beta\rangle$,

The hermitian conjugate of ∂_+

$$(\partial_+)^\dagger = \partial_-$$

yields

$$\langle \alpha, \beta | \partial_- \partial_+ | \alpha, \beta \rangle \geq 0$$

and

$$\langle \alpha, \beta | \partial_+ \partial_- | \alpha, \beta \rangle \geq 0$$

Hence also

$$\langle \alpha, \beta | \frac{1}{2} (\partial_+ \partial_- + \partial_- \partial_+) | \alpha, \beta \rangle \geq 0$$

$$\frac{1}{2} (\partial_+ \partial_- + \partial_- \partial_+) = \partial_x^2 + \partial_y^2 = \partial^2 - \partial_z^2$$

$$\hookrightarrow \quad \alpha - \beta^2 \geq 0$$

Hence the chain of eigenstates
has to terminate in both directions and there
are minimal and maximal values of β.

Using

$$J_+ J_- = J^2 - J_z^2 + J_z$$

and

$$J_- J_+ = J^2 - J_z^2 - J_z$$

One obtains for the minimal β

$$0 = \langle \alpha, \beta_{min} | J^+ J^- | \alpha, \beta_{min} \rangle$$

$$\hookrightarrow \quad \alpha - \beta_{min}^2 + \beta_{min} = 0$$

And likeswise

$$\alpha - \beta_{max}^2 - \beta_{max} = 0$$

Accordingly one finds

$$-\beta_{min} = +\beta_{max} = \alpha$$

$$\hookrightarrow \quad J^2 | l, m \rangle = l(l+1) | l, m \rangle$$

$$J_z | l, m \rangle = m | l, m \rangle$$

l has to be half-integer and positive and
m in [-l, l].

Examples:

$$l = \frac{1}{2} : \qquad J_i = \frac{1}{2}\sigma_i$$

whith the Pauli matrices

$$l = 1 :$$

$$\sigma_t = \begin{pmatrix} 0 & -i & \\ i & 0 & \\ & & 0 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & & \\ & 0 & i \\ & -i & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & & i \\ & 0 & \\ -i & & 0 \end{pmatrix}$$

As an example of a reducible representation, consider a rank 2 tensor.

This contains representations with l = 2,1,0.

Total number of degrees of freedom is

5+ 3+ 1 = 9