Vectors and tensors

We have seen that the coordinates transform according to

$$x'^{\beta} = \Lambda^{\beta}{}_{\alpha} x^{\alpha} \; ; \qquad \eta_{\alpha\beta} \Lambda^{\alpha}{}_{\gamma} \Lambda^{\beta}{}_{\delta} = \eta_{\gamma\delta}$$

In general any quantity that transforms as

$$V'^{\beta} = \Lambda^{\beta}{}_{\alpha} V^{\alpha}$$

is called a contravariant vector.

In contrast there are also covariant vectors that transform as

$$U'_{\beta} = \Lambda_{\beta}{}^{\alpha} U_{\alpha} \quad \text{where}$$

$$\Lambda_{\beta}{}^{\alpha} = \eta_{\beta\delta} \eta^{\alpha\delta} \Lambda^{\gamma}{}_{\delta}$$

and $\quad \eta^{\alpha\beta} = (\eta_{\alpha\beta})^{-1} \equiv \eta_{\alpha\beta}$ in flat space

Notice that these Lorentz trafos are inverse to each other in the sense

$$\Lambda^\alpha{}_\beta \Lambda_\alpha{}^\gamma = \Lambda^\alpha{}_\beta \eta_{\alpha\mu} \eta^{\gamma\nu} \Lambda^\mu{}_\nu = \eta_{\beta\nu} \eta^{\gamma\nu}$$
$$= \delta^\gamma{}_\beta$$

It follows that the contraction of contravariant and covariant vectors are scalars (invariant).

$$U'_\alpha V'^\alpha = \underbrace{\Lambda_\lambda{}^\mu \Lambda^\lambda{}_\nu}_{\delta^\mu{}_\nu} U_\mu V^\nu = U_\mu V^\mu$$

Contravariant vectors can be transformed into covariant vectors using the metric

$$U_\alpha = \eta_{\alpha\beta} U^\beta$$

$$U'_\alpha = \eta_{\alpha\beta} \overbrace{\Lambda^\beta{}_\gamma U^\gamma}^{U'^\beta}$$
$$= \Lambda_\alpha{}^\nu \underbrace{\Lambda^\mu{}_\nu \eta_{\mu\beta} \Lambda^\beta{}_\gamma}_{\eta_{\nu\gamma}} U^\gamma$$

$$= \Lambda_\alpha{}^\nu (\eta_{\nu\gamma} U^\gamma) = \Lambda_\alpha{}^\nu U_\nu$$

Another example for a scalar is the line element

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

The differential dx^μ

is contraviariant. $x'^\mu = \Lambda^\mu{}_\nu x^\nu$

The derivative operator is covariant:

$$\frac{d}{dx'^\mu} = \frac{dx^\nu}{dx'^\mu}\frac{d}{dx^\nu} = \Lambda_\mu{}^\nu \frac{d}{dx^\nu}$$

$$\frac{d}{dx^\mu} = \partial_\mu$$

$$\partial'_\mu = \Lambda_\mu{}^\nu \partial_\nu$$

We can use this to contruct covariant vector fields out of scalar fields.

$\varphi(x^\mu)$; $\partial_\mu \varphi(x^\mu)$ $\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$

scalar cov. vector scalar

$x^\mu \partial_\mu \varphi$

scalar.

There are also quantities with several indices. These are tensors. The indices can be either covariant or contravariant.

$$T^{\alpha\beta\prime} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu T^{\mu\nu}$$

New tensors can be constructed from old ones via

A) linear combination $T^{\alpha}{}_{\beta} = a A^{\alpha}_{\beta} + b B^{\alpha}_{\beta}$

B) (outer) product $T^{\alpha}{}_{\beta} = V^{\alpha} \cdot U_{\beta}$

C) contraction $A^{\alpha} = T^{\alpha\beta}{}_{\beta}$

D) differentiation $T_{\alpha}{}^{\beta} = \partial_{\alpha} A^{\beta}(x^{\mu})$

There are a couple of special tensors:

(i) Minkowski metric

$$\eta'_{\mu\nu} = \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu} \eta_{\alpha\beta} = \eta_{\mu\nu}$$

(ii) The levi civita tensor

$$\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} 1 & \text{for } \alpha\beta\gamma\delta \text{ even permutations of } 0123 \\ -1 & \text{for odd perm.} \\ 0 & \text{otherwise} \end{cases}$$

$$\epsilon'^{\alpha\beta\gamma\delta} \overset{!}{=} \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu} \Lambda^{\gamma}{}_{\lambda} \Lambda^{\delta}{}_{\kappa} \epsilon^{\mu\nu\lambda\kappa}$$

$$\dot{=} \ \epsilon^{\alpha\beta\gamma\delta} (\det \Lambda) = \epsilon^{\alpha\beta\gamma\delta}$$

For the proof notice that the RHS is totally antisymmetric and that for $\alpha\beta\gamma\delta = 0123$ the expression is the determinant.

(iii) zero tensors

comments:

Since $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$ are tensors, they can be used to lower and raise indices.

Notice that the order of the indices is important

$$T^{\mu\nu} \neq T^{\nu\mu}$$ in general

$$T^{\mu}{}_{\nu} \neq T_{\nu}{}^{\mu}$$

Remember that $\Lambda^{\alpha}{}_{\beta}$ and $\Lambda_{\alpha}{}^{\beta}$ are inverse of each other

[Carroll is using

$$\Lambda^{\mu}{}_{\mu'} \text{ and } \Lambda^{\mu'}{}_{\mu}$$]

Electrodynamics

Maxwell's equations are:

$$\vec{\nabla} \cdot \vec{E} = \rho \quad (\text{density})$$

$$\vec{\nabla} \times \vec{B} = \partial \vec{E} / \partial t + \vec{J} \quad (\text{current})$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

In order to check the transformation properties we construct the following tensor:

$F^{\mu\nu}$: (field strength)

$$F^{ij} = \epsilon^{ijk} B_k$$

$$F^{0i} = -F^{i0} = E_i$$

$$F^{\mu\nu} = -F^{\nu\mu}$$

J^{μ}: (four current)

$$J^0 = \rho$$

$$J^i = \vec{J}^i$$

The Maxwell equations then read

$$\frac{d}{dx^\alpha} F^{\alpha\beta} = -j^\beta$$

$$\epsilon^{\alpha\beta\gamma\delta} \frac{d}{dx^\beta} F_{\gamma\delta} = 0$$

$$\epsilon_\alpha{}^\beta{}_{\gamma\delta}\, \partial_\beta F^{\gamma\delta} = 0$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & \vec{E} & & \\ -\vec{E} & 0 & B_3 & -B_2 \\ & & 0 & B_1 \\ & & & 0 \end{pmatrix} \quad F_{\mu\nu} = \begin{pmatrix} & -\vec{E} & & \\ \vec{E} & 0 & B_3 & -B_2 \\ & & 0 & B \\ & & & 0 \end{pmatrix}$$

$$F'^{\mu\nu} = \Lambda^\nu{}_\beta \Lambda^\mu{}_\alpha F^{\alpha\beta}$$

$$\Lambda : \begin{pmatrix} 1 & \\ & R \end{pmatrix}$$

$$\vec{E}' = R\vec{E}$$

$$\vec{B}' = R\vec{B}\,? \quad \rightarrow \text{exercise}$$