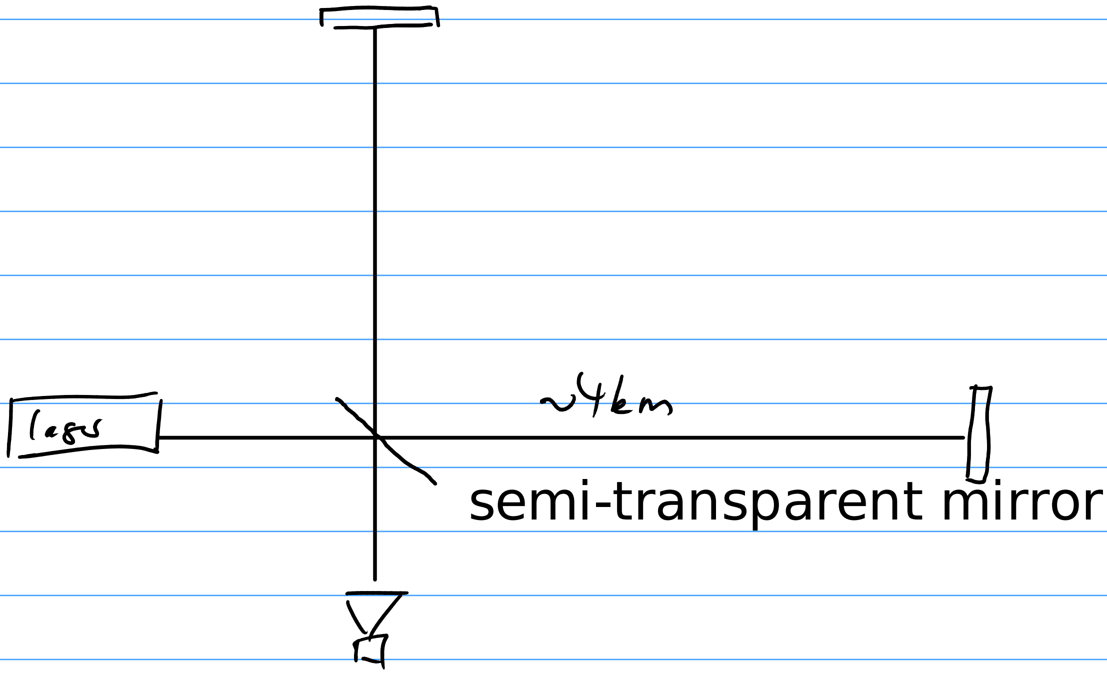
Laser interferometers:



comments:

- one wants to observe the impact of the gravitational wave on the interference patters

GW in TT gauge

$$h_{\mu\nu} = \begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix} e^{i k_\mu x^\mu} + c.c.$$

geodesic equation:

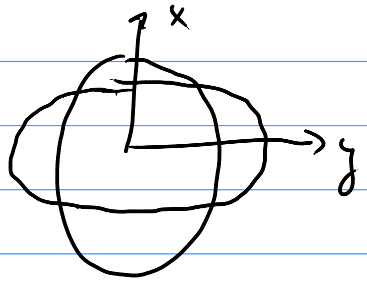
$$\frac{\partial^2 x^\mu}{(\partial\lambda)^2} + \Gamma^\mu_{\nu\kappa} \frac{\partial x^\nu}{\partial\lambda} \frac{\partial x^\kappa}{\partial\lambda} = 0$$

$$\frac{\partial x^\mu}{\partial\lambda} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \Gamma^\mu{}_{00} = 0 \quad \text{in TT gauge}$$

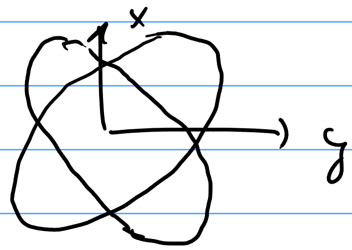
mirrors are static!

However, the runtime of the light changes due to the metric.

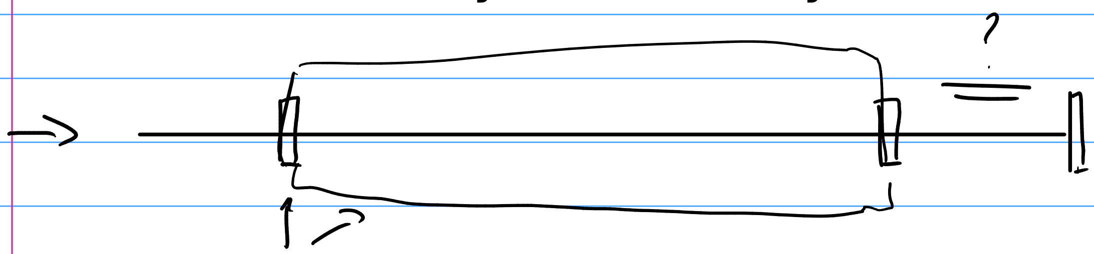
h_+ :



$h_\times$:



- most ground based experiments, a cavity is used -> Fabry-Pérot cavity



more semi-transparent mirrors
-> light bounces a couple of times in the cavity before it leaves
-> effective enhancement of the arm-length.

$$400\,\text{km} \simeq \frac{400\,\text{km}}{3\cdot 10^8\,\text{m/s}} \simeq \frac{4}{3}\cdot 10^{-3}\ \text{sec}$$

$$\simeq \frac{3}{4}\cdot 10^{3}\ \text{Hz}$$

What is the amplitude of fluctuations we could hope for?

$$e_{\mu\nu} = \frac{4G}{r} \int d^3x \; S_{\mu\nu}(x,t)$$

$$e_{\mu\nu} \sim \frac{4G}{r} \; m R^2 \omega_p^2$$

Putting typical numbers

$m \sim 30 M_\odot$, $r \simeq 400 \, Mpc$

$\omega_p \sim kHz$

$R \sim 10 \, km$

$\rightarrow \quad h \sim 10^{-20}$; $R\omega_p \simeq \frac{1}{30}$

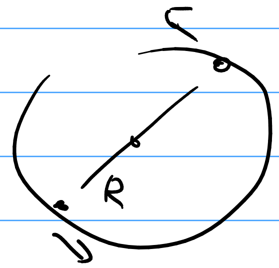
If the sensitivity is improved by a factor 2, then one can observe events that are a factor 2 further away! -> observed volume is enhanced by a factor 8!

time dependence of the signal:

$$P \sim \frac{2G\omega^6}{5}\left[D_{ij} D^*_{ij} - \frac{1}{3} |D_{ii}|^2 \right]$$

$$\sim \frac{128 G \omega^6}{5} m^2 R^4$$

Now we can use Keplers law:



$$m a = m R \omega^2$$

$$\overset{!}{=} G \frac{m^2}{(2R)^2}$$

$$\omega^2 R^3 = \frac{Gm}{4} \rightarrow R \propto \left(\frac{Gm}{\omega^2} \right)^{1/3}$$

$$P_{GW} = G \omega^6 m^2 R^4 = G^{7/3} \omega^{10/3} m^{10/3}$$

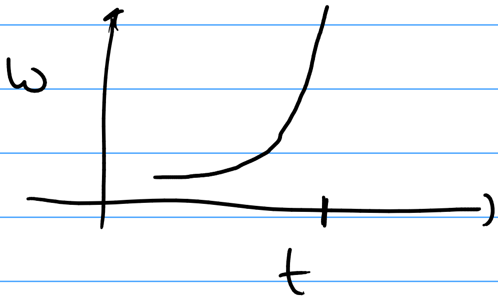
$$E_b = \frac{1}{2} m R^2 \omega^2 \propto G^{2/3} m^{5/3} \omega^{2/3}$$

$$\dot{E}_b = \dot{\omega} G^{2/3} m^{5/3} \omega^{-1/3} \overset{!}{=} P_{GW}$$

$$= G^{7/3} \omega^{10/3} m^{10/3}$$

$$\omega \sim (Gm)^{-5/8} t^{-3/8}$$

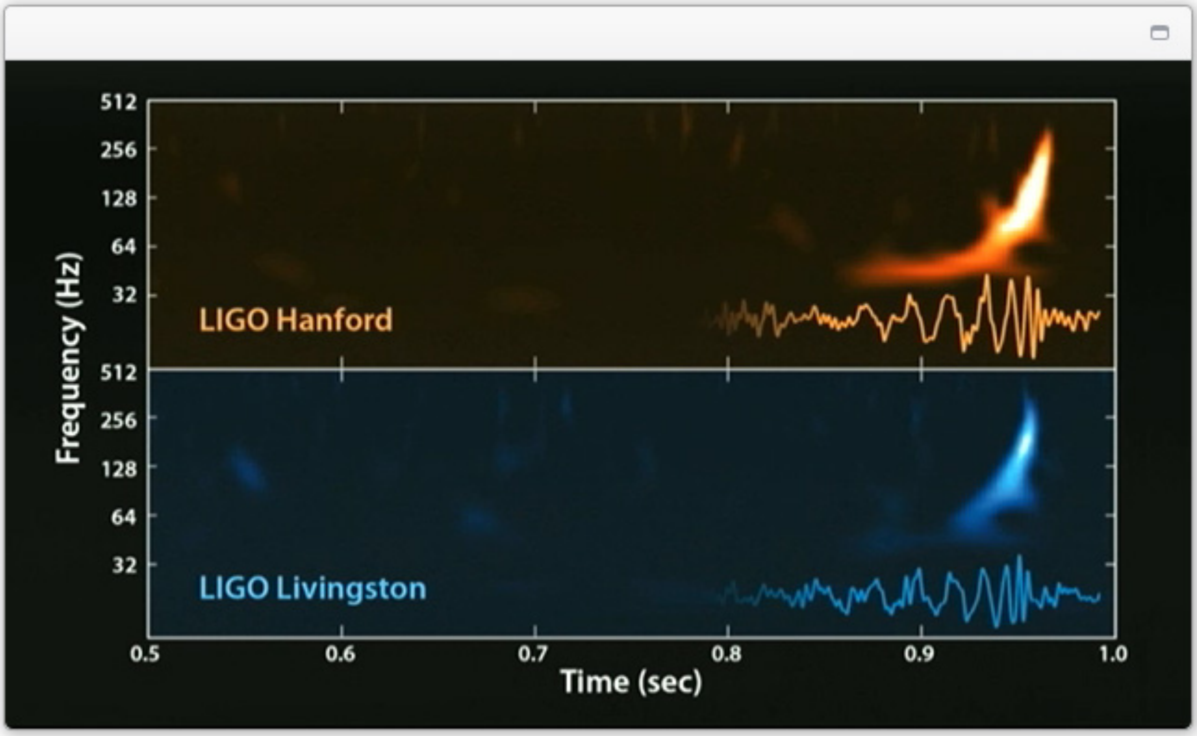
the merger happens at t=0.



We discussed the simplest case of two equal masses on a spherical oribit.

In general, there is a combination of the two masses showing up in this relation, which is called the 'chirp mass'.
(see exercise)

LIGO Update on the Search for Gravitational Waves



Cosmology

Cosmology starts from the cosmological principle: The Universe is isotropic and homogenous on very large scales.
In particular, the metric is supposed to be homogenous and isotropic on very large scales.

The 3D Euclidean metric is homogenous and isotropic.

$$ds^2 = d\vec{x} \cdot d\vec{x} \quad ; \quad g_{ij} = \mathbb{1}_{3\times3}$$

homogeniety: $\vec{x} \to \vec{x} + \vec{a} \quad ; \; d\vec{x} \to d\vec{x}$

isotropy:

$$\vec{x} \to O \cdot \vec{x} \quad ; \quad d\vec{x} \to O \cdot d\vec{x}$$

Interestingly, the cosmological priciple is consistent with 3D curvature:

Consider:

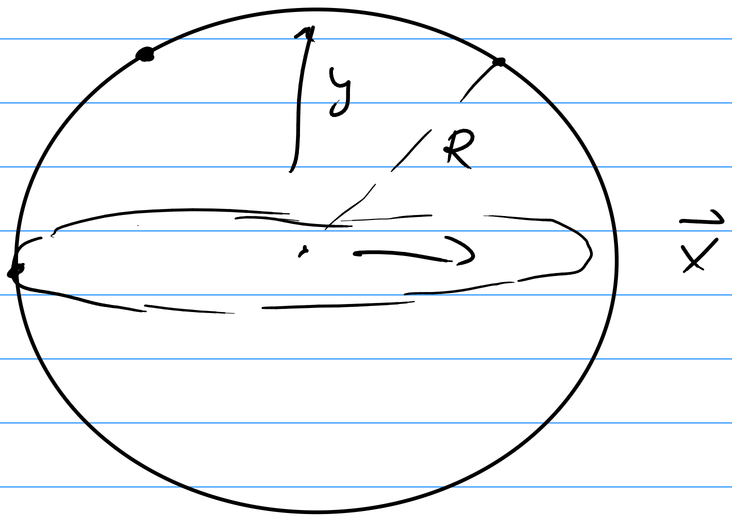
$$ds^2 = \underbrace{dy^2}_{+1} + \underbrace{d\vec{x} \cdot d\vec{x}}_{3D}$$

Constraint: $y^2 + x^2 = R^2$

$$2\,dy\,y + 2\vec{x}\,d\vec{x} = 0$$

$$dy^2 = \frac{(\vec{x} \cdot d\vec{x})^2}{y^2} = \frac{(\vec{x} \cdot d\vec{x})^2}{R^2 - x^2}$$

$$O\begin{pmatrix} y \\ \vec{x} \end{pmatrix}$$



$SO(4)$

embedded metric:

$$ds^2 = d\vec{x} \cdot d\vec{x} + \frac{(\vec{x} \cdot d\vec{x})^2}{y^2} \quad / \quad y^2 = R^2 - x^2$$

In spherical coordinates:

$$ds^2 = \frac{R^2}{R^2 - r^2} dr^2 + r^2 d\Omega^2$$

Rescaling r leads to

$$ds^2 = \frac{1}{1 - kr^2} dr^2 + r^2 d\Omega^2$$

$$\boxed{k = +1} \quad ; \quad r^2 < 1$$

This space still has 6 isometries, which are the 6 rotations in 4D Euclidean space.

$$\begin{pmatrix} y \\ \vec{x} \end{pmatrix} \to O \begin{pmatrix} y \\ \vec{x} \end{pmatrix} \qquad O \in SO(4)$$

Obviously, for k=0 one obtains 3D Euclidean flat space. One can obtain k=-1 when one embeds into the 4D space with Minkowski signature

$$ds^2 = -dy^2 + d\vec{x} \cdot d\vec{x}$$

and the constraint

$$R^2 = y^2 - x^2$$

If the 4D metric of the Universe is not assumed to be static,
the most general metric of the Universe is then the Friedmann-Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$$

scale factor.

Energy-momentum tensor of the Universe

For the energy-momentum tensor, we consider a fluid

$$T^{\mu\nu} = \int d^4p \, \gamma_g \, p^\mu p^\nu \, f(p^\mu u_\mu / T) \, \delta(p^2 - m^2)$$

p is the momentum
f(p) is the particle-distribution function

u is the plasma four-vector.

$$\sqrt{g} = a^3 \qquad u^\mu u_\mu = -1 \qquad (g_{00} = 1)$$

The energy-momentum tensor is of the form

$$T^{\mu\nu} = u^\mu u^\nu w a^3 + g^{\mu\nu} P \cdot a^3$$

P(t) = pressure
w(t) = enthalpy
e(t) = energy density

$$w = P + e \Rightarrow$$

$$e = \int d^4p \; \delta \cdot f \cdot (u^\mu p_\mu)^2$$

$$4P - w = -\int d^4p \; \delta \; f \cdot (m^2)$$

In the plasma frame: $u^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$e = \int \frac{d^3\vec{p}}{2E} f(E/T)\, E^2$$

$$P = \int \frac{d^3\vec{p}}{2E} f(E/T)\, p^2/3$$

Energy-momentum conservation:

$$D_\mu T^{\mu\nu} = 0 \quad \Rightarrow$$

$$\partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\alpha} T^{\alpha\nu} + \Gamma^\nu_{\mu\beta} T^{\mu\beta}$$

$$\left[\quad \frac{\partial g_{00}}{\partial x^\mu} = 0 \;; \quad \frac{\partial g_{ij}}{\partial t} = 2 \cdot \frac{\dot a}{a} g_{ij} \;, \quad \frac{\partial g_{ij}}{\partial x^k} \overset{\overset{k}{\downarrow}}{\neq} 0 \right]$$

$$\hookrightarrow \quad \Gamma^0_{ij} = \dot a a \cdot \delta_{ij}$$

$$\Gamma^i_{0j} = \frac{\dot a}{a} \delta^i_j$$

$$\Gamma^i_{jk} \neq 0 \quad (\text{curvature})$$

Feeding this into the Ansatz for the energy-momentum tensor yields:

$$\boxed{\begin{aligned} a^3 \frac{\partial p}{\partial t} &= \frac{\partial}{\partial t}\left(a^3 (e + \underline{p})\right) \\ \Leftrightarrow \quad \frac{\partial}{\partial t}(e a^3) &= -3 \underline{p} a^3 \frac{\dot a}{a} \end{aligned}}$$

Moreover, the Einstein equations look

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3} e$$

This is the Friedmann equation.

$\left(\frac{\dot{a}}{a}\right) \equiv H$ is the Hubble parameter.

The other equations are either trivial or equivalent to the energy-momentum conservation.

Equation of state

Consider a relatively simple relationship between the energy density e and the pressure P

$$P \propto e \, ; \quad P = c \cdot e$$

c is called the equation of state parameter and often also denoted ω (like enthalpy)

For example: radiation = massless particles

$$m^2 = 0 \Rightarrow \quad \omega = 4P \Rightarrow \quad e = 3P$$

$$\hookrightarrow \boxed{c = \frac{1}{3}}$$

non-relativistic matter: $m^2 \gg T$

$$\underline{P} = \int \frac{d^3p}{2E}\, p^2 f(E/T) \ll$$

$$e = \int \frac{d^3p}{2E}\, E^2 f(E/T)$$

$$E \sim m + O(p^2/m)$$

$$\hookrightarrow p \approx 0 \cdot e \quad \hookrightarrow \boxed{c = 0}$$

cosmological constant:

One can always add a constant term
to the matter Lagrangian

$$S_{matter} \ni \int d^4x \sqrt{g}\, \underline{\Lambda}$$

$$\hookrightarrow T^{\mu\nu} \propto g^{\mu\nu} \underline{\Lambda}$$

This will leave the quation of motion of matter
unchanged, but the Einstein equation
obtains the additional term in the
energy-momentum tensor.

$$\omega = 0 \Rightarrow p = -e \Rightarrow \boxed{c = -1}$$