Comments on the Schwarzschild metric

$$-ds^2 = d\tau^2 = \left[1 - \frac{2MG}{r}\right] dt^2 - \left[1 - \frac{2MG}{r}\right]^{-1} dr^2 - r^2 d\Omega^2$$

The metric has singularities at r=0 and r = 2MG.

The singularity at r=0 is expected. This is analogous to the singularity in the Coulomb potential.

But the singularity at r=2MG is kind of unexpected.

This is called the Schwarzschild radius

$$r_c = 2MG$$

A minimal requirement for a singularity not to be harmful is that all scalars are regular:

$$R, \quad R_{\mu\nu} T^{\mu\nu}, \quad R_{\mu\nu} R^{\mu\nu}, \quad R_{\mu\nu\lambda\kappa} R^{\mu\nu\lambda\kappa}$$

$$R_{\mu\nu} g^{\mu\nu}$$

In our case

$$T_{\mu\nu} = 0 \rightarrow G_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$$

but

$$R_{\mu\nu\kappa\lambda} R^{\mu\nu\kappa\lambda}$$

does not vanish by default and it turns out that it is not singular at r=2MG.

There are actually coordinate systems (e.g. Kruskal coordinates) where the metric is regular at r=2MG.

$$d\tau^2 = \left(\frac{32 G^3 M^3}{r T^2} \right) \exp\left(\frac{-r}{2GM} \right) (dt^2 - dv^2) - r^2 d\Omega^2$$

with T an arbitrary parameter.

So even though the metric is not pathological at the Schwarzschild radius (=horizon), it has some physical implications:

Consider the motion of a light along a radial path:

$$d\tau = -ds^2 = 0$$

$(d\varphi = d\theta = 0 \rightarrow$ radial motion$)$

$$\hookrightarrow \left(\frac{dr}{dt}\right)^2 = \left(1 - \frac{2GM}{r}\right)^2$$

$$\frac{dr}{dt} = \pm \left(1 - \frac{2GM}{r}\right)$$

If you solve this, one finds that for

$$t \rightarrow \infty \quad ; \quad r \rightarrow r_c = 2MG$$

So in this coordinate system, light will never reach the horizon.

If you would calculate the proper time of somebody falling into the BH, then he would only reach finite proper time when t -> ∞.

For example, if a rocket would approach the horizon, any light signal from the rocket would more and more redshift.

It also implies that anything within the horizon cannot get to an observer far away from the BH in finite time.

So if one is interested in what happens close to the horizon, one better uses more local coordinates (e.g. Kruskal, Eddington-Finkelstein)

So the standard form only captures
part of the manifold that can in principle be
described by the Einstein equations
+ geodesics -> extended coordinate systems.

Besides the Schwarzschild solution
there is actually a zoo of BH solutions:

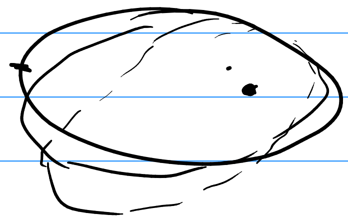
BH with charge/angular momentum.
In higher dimensions there are even more
solutions and depending on the symmetries
there are also black branes, and so on.

Classical tests of GR

Einstein suggested at the time three different tests of GR

- bending of light by the sun
- anomalous perihelion rotation of Mercury
- redshift of light in gravitational potential

Other early tests:
- radio echo delay (Shapiro)
- precession of gyroscopes in the earth gravitational potential
- gravitational waves:
    - direct
    - indirect (pulsars)



Geodesics in the Schwarzschild background

$$\frac{\partial^2 x^\mu}{(\partial\lambda)^2} + \Gamma^\mu{}_{\nu\kappa}\frac{\partial x^\nu}{\partial\lambda}\frac{\partial x^\kappa}{\partial\lambda} = 0$$

Since the problem is isotropic, we can constrain the path to $\Theta = \frac{\pi}{2}$

$$d\tau^2 = B(r)\,dt^2 - A(r)\,dr^2 - r^2 d\varphi^2$$

$$\frac{d^2 r}{d\lambda^2} + \frac{A'}{2A}\left(\frac{dr}{\partial\lambda}\right)^2 - \frac{r}{A}\left(\frac{d\varphi}{\partial\lambda}\right)^2 + \frac{B'}{2A}\left(\frac{dt}{d\lambda}\right)^2 = 0$$

$$\frac{d^2\varphi}{\partial\lambda^2} + \frac{2}{r}\frac{\partial\varphi}{\partial\lambda}\frac{\partial r}{\partial\lambda} = 0$$

$$\frac{\partial^2 t}{(\partial\lambda)^2} + \frac{B'}{B}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0$$

The last two equations can be recast as:

$$\frac{\partial\varphi}{\partial\lambda}r^2 = \text{const}$$

$$\frac{\partial t}{\partial\lambda}B(r) = \text{const.}$$

We still have the freedom to rescale λ with a constant, and we choose:

$$\frac{dt}{d\lambda} = \frac{1}{B(r)}$$

$$r^2 \frac{d\varphi}{d\lambda} = J$$

Notice that this law resembles Keplers law.

The last remaining equation then reads:

$$0 = \frac{d^2r}{(d\lambda)^2} + \frac{A'}{2A}\left(\frac{dr}{d\lambda}\right)^2 - \frac{J^2}{r^3 A} + \frac{B'}{2AB^2}$$

which is equivalent to

$$\frac{d}{d\lambda}\left[A \left(\frac{dr}{d\lambda}\right)^2 + \frac{\gamma^2}{r^2} - \frac{1}{B} \right] = 0$$

or

$$A \left(\frac{dr}{d\lambda}\right)^2 + \frac{\gamma^2}{r^2} - \frac{1}{B} = -E$$

where E quantifies the total energy.

One can determine the proper time of the particle:

$$d\tau^2 = E \, d\lambda^2$$

Such that $E=0$ for light
$E>0$ for massive particles

We can now eliminate the parameter λ in favor of time everywhere:

$$r^2 \frac{\partial \varphi}{\partial t} = \gamma B(r)$$

$$\frac{A}{B^2}\left(\frac{dr}{\partial t}\right)^2 + \frac{\gamma^2}{r^2} - \frac{1}{B} = -E$$

$$d\tau^2 = E B^2 dt^2$$

In the limit of a slow particle in a weak gravitational field, one gets the Newtonian limit:

$$B - 1 \simeq -(A-1) \simeq 2\Phi$$

$$r^2 \frac{\partial \varphi}{\partial t} = \gamma$$

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{\gamma^2}{2r^2} + \Phi = \frac{1-E}{2} \simeq \text{energy per mass}$$

If we are not interested in the dynamics but just in the shape of the orbits, we can also eliminate the time :

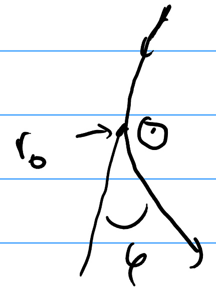
$$\left(\frac{A}{r^4}\right)\left(\frac{dr}{d\varphi}\right)^2 + \frac{1}{r^2} - \frac{1}{J^2 B} = -\frac{E}{J^2}$$

Remember:

$$\left(\frac{A}{r^4}\right)\left(\frac{dr}{d\varphi}\right)^2 + \frac{1}{r^2} - \frac{1}{J^2 B} = -\frac{E}{J^2}$$

Deflection of light by the sun

For light we have seen that E=0
and in the point closest to the sun:



$$\left.\frac{\partial r}{\partial \varphi}\right|_{r=r_0} = 0 \implies J^2 = \frac{r_0^2}{B(r_0)}$$

Now we integrate $\frac{\partial r}{\partial \varphi}$ to obtain the deflection angle

$$\Delta\varphi \simeq \frac{4MG}{r_0} = \frac{2 r_c}{r_0}$$

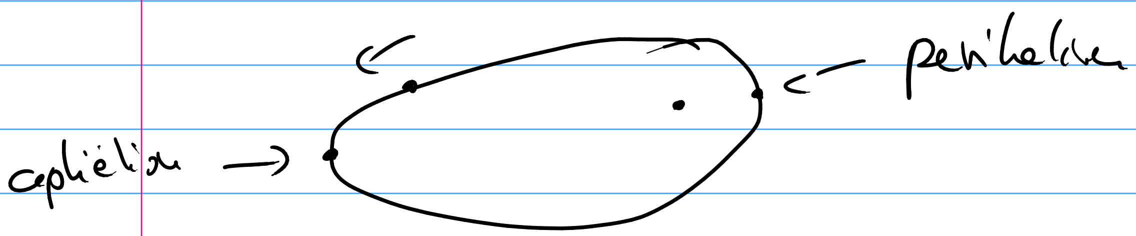
For the sun, the maximal deflection angle is
$(r_0 = r_\odot, \quad M = M_\odot)$

$$\Delta\varphi = 1.75'' \quad (\text{arcsec})$$

There have been several expeditions
to solar eclipses to measure this (starting from
1919) and GR was confirmed.

precession of Mercury perihelion

In Newton's mechanics, a single planet around
the sun describes an ellipse
with a static perihelion



However, there are several effects that
let the perihelion rotate:

- earth based astronomy (~5025''/century)
- 3 body dynamics (~530''/century)

Still there was an anomalous piece in the
perihelion roation ~ 43''/century

This discrepancy was nicely explained by GR.

Radar echo delay

One can test GR by sending radio signals to the inner planets and observe the reflections Since in GR the signal does not go on a straight line (and other effects) there is a delay in the arrival time of the reflected signal.

This was suggested by Shapiro and tested in the 1960s, again in agreement with GR.