Gauge freedom

The Einstein equations

$$G^{\mu\nu} = -8\pi G T^{\mu\nu}$$

$G^{\mu\nu}$	Einstein tensor
G	Newton's constant
$T^{\mu\nu}$	energy-momentum tensor

are 10 equations and there are 10 degrees of freedom. However, the Einstein tensor fulfills the coservation law

$$\nabla_\mu G^{\mu\nu} = -8\pi G \nabla_\mu T^{\mu\nu} = 0$$

and hence the Einstein equations only encode 6 independent equations.

Hence the metric $g_{\mu\nu}$ is not unique. This is by no means surprising, since there are 4 coordinate transformations that leave the Einstein eq. invariant.

This is in full analogy to EM

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\partial_\mu F^{\mu\nu} = - J^\nu$$

This encodes only 3 = 4 - 1 equations since

$$\partial_\nu \partial_\mu F^{\mu\nu} = 0 = - \partial_\nu J^\nu$$

-> 1 gauge degree of freedom

$$A_\mu \to A_\mu + \partial_\mu \alpha$$

The problem can be removed by fixing the gauge first and then solving the equations of motion.

One common gauge choice is the harmonic gauge

$$\Gamma^\lambda \equiv g^{\mu\nu} \Gamma^\lambda{}_{\mu\nu} = 0$$

This gauge can always be reached because the constraint transforms under coordinate transformations as

$$\Gamma'^\lambda = g'^{\mu\nu} \Gamma'^\lambda{}_{\mu\nu} = \frac{\partial x'^\lambda}{\partial x^\rho} \Gamma^\rho{}_{\alpha\beta} g^{\alpha\beta} - g^{\rho\sigma} \frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\sigma}$$

For non-vanishing Γ^{λ} this can be solved for

$$g^{\rho\sigma} \frac{\partial x'^{\lambda}}{\partial x^{\rho} \partial x^{\sigma}} = \frac{\partial x'^{\lambda}}{\partial x^{\delta}} \Gamma^{\delta}{}_{\alpha\beta} g^{\alpha\beta}$$

$$\hookrightarrow \quad \Gamma'^{\lambda} = 0$$

Using the Christoffel symbols one finds

$$\Gamma^{\lambda} = -\sqrt{g}\, \partial_{x^{\kappa}} \left(\sqrt{g}\, g^{\lambda\kappa}\right)$$

and the harmonic gauge implies

$$\partial_{x^{\kappa}} \left(\sqrt{g}\, g^{\lambda\kappa}\right) = 0$$

Concerning the name:

A harmonic function is a function that fulfills

$$\Box \varphi = 0$$

$$\hookrightarrow \quad g^{\lambda\kappa} \frac{\partial^2 \varphi}{\partial x^{\lambda} \partial x^{\kappa}} - \Gamma^{\lambda} \frac{\partial \varphi}{\partial x^{\lambda}} = 0$$

If you understand the coordinates as scalars, then one would get

$$\Box x^{\mu} = 0$$

and the coordinates are harmonic functions.

Isotropic solutions

Consider an isotropic and static Ansatz for the metric

$$-ds^2 = d\tau^2 = F(r)\,dt^2 - 2E(r)\,dt\,d\vec{x}\cdot\vec{x}$$

$$- D(r)\,(\vec{x}\cdot d\vec{x})^2 - C(r)\,d\vec{x}^2$$

or in spherical coordinates

$$d\tau^2 = F(r)\,dt^2 - 2r\,E(r)\,dt\,dr$$

$$- r^2 D(r)\,dr^2 - C(r)$$

$$\times (dr^2 + \underbrace{r^2 d\Theta^2 + r^2 \sin^2\Theta\, d\varphi^2}_{r^2 d\Omega^2})$$

We are still free to change the coordinate system as long as it abides to isotropy, e.g.

$$t' = t + \Phi(r)$$

This can be used to eliminate the term $dt\,dr$

$$dt' = dt + dr\ \partial_r \Phi$$

$$\hookrightarrow \; d\tau^2 = F(r)\,dt'^2 - G(r)\,dr^2$$
$$- C(r)\,(dr^2 + r^2 d\Omega^2)$$

where $\quad G(r) = r^2\left(D(r) + E^2(r)/F(r)\right)$

Redefining the radius $\qquad (f' \doteq \partial_r f)$

$$r = f(r') \;\rightarrow\; dr = dr'\, f'$$

$$d\tau^2 = F(r)\,dt'^2 - (G + C)\, f'^2\, dr'^2$$
$$+ C(r)\, f^2\, d\Omega^2$$

for $r'^2 = C(r) r^2$ one obtains the standard form

$$d\tau^2 = B(r)\,dt^2 - A\,dr^2 \underset{\substack{\uparrow \\ 1}}{-} r^2 d\Omega^2$$

Alternatively, one can go to the isotropic form

$$d\tau^2 = A(r)\,dt'^2 - J(r)\,(dr^2 + r^2 d\Omega^2)$$

Finally, there is also the harmonic gauge:

$$d\tau^2 = B(r)\,dt^2 - \frac{r^2}{R^2}\,d\vec{X}^2 - \left[\frac{A(r)}{R^2 R'^2} - \frac{r^2}{R^4}\right](dX\cdot X)^2$$

with R fulfilling

$$\frac{d}{dr}\left(r^2\sqrt{B/A}\,\frac{dR}{r}\right) = 2\sqrt{AB}\,R$$

In the following we will use the standard form

$$d\tau^2 = B(r)\,dt^2 - A(r)\,dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

The metric is diagonal and the inverse $g^{\mu\nu}$ is easily constructed. Also the Christoffel symbols and the Riemann tensor are straight forward. For Ricci tensor one finds

$$R_{rr} = \frac{B''}{2B} - \frac{1}{4}\left(\frac{B'}{B}\right)\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{r}\,\frac{A'}{A}$$

$$R_{\theta\theta} = -1 + \frac{r}{2A}\left(-\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{A}$$

$$R_{\varphi\varphi} = \sin^2\theta\, R_{\theta\theta}$$

$$R_{tt} = -\frac{B''}{2A} + \frac{1}{4}\left(\frac{B'}{A}\right)\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{r}\left(\frac{B'}{A}\right)$$

$$R_{\mu\nu} = 0 \quad \text{for } \mu \neq \nu$$

The relation $R_{\varphi\varphi} = \sin^2\theta\, R_{\theta\theta}$ is a consequence of isotropy.

Schwarzschild metric

The Schwarzschild solution assumes a mass in teh origin (r = 0) and empty space otherwise.

$$R_{\mu\nu} = 0$$

$$\hookrightarrow R_{rr} = R_{\theta\theta} = R_{tt} = 0$$

$$\frac{R_{rr}}{A} + \frac{R_{tt}}{B} = -\frac{1}{rA}\left(\frac{A'}{A} + \frac{B'}{B}\right)$$

$$\hookrightarrow \frac{A'}{A} + \frac{B'}{B} = 0$$

$$\hookrightarrow A \cdot B = \text{const.}$$

We want the metric to reduce to the flat metric at infinity

$$A \xrightarrow{r \to \infty} 1$$

$$B \xrightarrow{r \to \infty} 1$$

$$\hookrightarrow \boxed{A = \frac{1}{B}}$$

Using this in $R_{\theta\theta}$ one finds

$$R_{\theta\theta} = -1 + B' r + B$$

and

$$R_{rr} = \frac{B''}{2B} + \frac{B'}{rB} = \frac{R'_{\theta\theta}}{2rB}$$

So finally:

$$R_{\theta\theta} = -1 + \partial_r (Br) = 0$$

$$rB = r + \text{const}$$

$$B = 1 + \frac{\text{const}}{r}$$

The constant can be fixed by the taking the Newtonian limit

$$g_{00} \simeq -(1 + 2\Phi)$$

and $\Phi = -MG/r$

So the full metric reads

$$\boxed{d\tau^2 = \left[1 - \frac{2MG}{r}\right] dt^2 - \left[1 - \frac{2MG}{r}\right]^{-1} dr^2 - r^2 d\Omega^2}$$