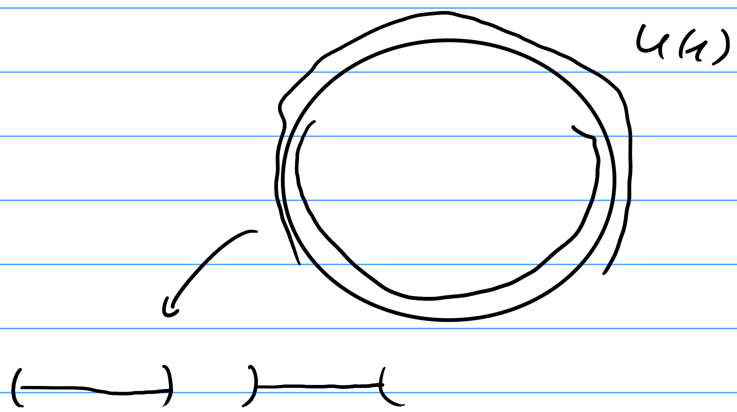
Manifolds

A manifold is a space (set of points) that is homeomorphic to $\mathbb{R}^n$.

This means, there is a map of open neighbourhoods in M to $\mathbb{R}^n$

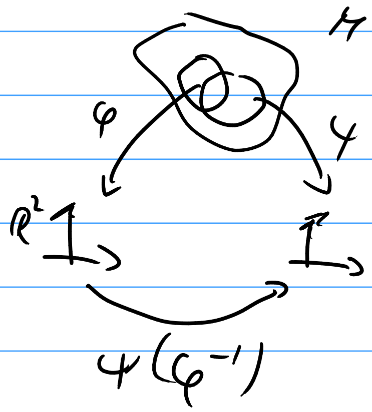
The map is called a chart.

Sometimes several charts are needed to cover the full manifold.



(homeomorphic -> bijective)

For our purpose, charts need to be compatible in their overlap, meaning their relationship is differentiable.



The manifold is called differentiable if these relationships between charts are.

On top, we will have a metric on the manifold that can be used to measure distances and volumes.

$$ds^2 = dx^\mu dx^\nu g_{\mu\nu} \rightarrow \quad dV = d^4x \sqrt{g}$$

Tangent spaces:

In every point of the manifold, a tangent space can be constructed with the basis dx^μ.

The most general element in the tangent space is then

$$V = V_\mu dx^\mu$$

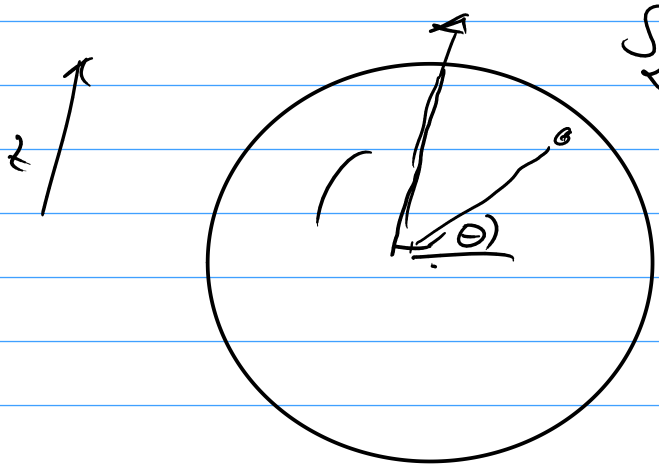
Notice that in principle the tangent spaces in different points are unrelated entitites

(x, y)

$$T_x M \quad \leftrightarrow \quad T_y M$$

Embeddings:

One way of constructing a "non-flat" manifold is by embedding it into higher dimensions:



S_2 2D sphere

The 2D sphere is an embedding into Euclidean 3D:

$$ds^2 = dx^2 + dy^2 + dz^2$$

And the sphere fulfills the constraint

$$x^2 + y^2 + z^2 = L^2$$

The induced metric can then be calculated by eliminating one coordinate:

$$0 = 2x\,dx + 2y\,dy + 2z\,dz$$

$$\hookrightarrow \quad ds^2 = dx^2 + dy^2 + dz^2 =$$

$$= dx^2 + dy^2 + \frac{(x\,dx + y\,dy)^2}{R^2 - x^2 - y^2} = dx^\mu g_{\mu\nu}\, dx^\nu$$

$$g_{\mu\nu} = \begin{pmatrix} 1 + \frac{x^2}{z^2} & \frac{xy}{z^2} \\ \frac{xy}{z^2} & 1 + \frac{y^2}{z^2} \end{pmatrix}$$

Or in general, starting from a space with coordinates

$$x^\mu \quad (\mu = 0..D)$$

and a metric $g(x)$. Using a subspace that is parameterized by some constraints

$$y^i \quad (i = 0 \ldots d < D)$$

$$x^\mu(y^i) \qquad \left[\; z = \sqrt{R^2 - x^2 - y^2},\ x = x,\ y = y \;\right]$$

one can construct the embedded metric as

$$ds_d^2 = dy^i dy^j \frac{dx^\mu}{dy^i} \frac{dx^\nu}{dy^j} g_{\mu\nu} \equiv dy^i dy^j \gamma_{ij}$$

spherical coordinates:

$$
\begin{aligned}
x &= L \cdot \cos\theta \cos\varphi \qquad & \theta \in [0, \pi] \\
y &= L \cdot \cos\theta \sin\varphi \qquad & \varphi \in [0, 2\pi] \\
z &= L \cdot \sin\theta
\end{aligned}
$$

This fulfills the constraint:

$$x^2 + y^2 + z^2 = L^2$$

And the induced metric is:

$$
\begin{aligned}
\frac{dx}{d\theta} &= -L \cdot \sin\theta \cos\varphi \qquad & \frac{dx}{d\varphi} &= -L \cdot \cos\theta \cdot \sin\varphi \\
\frac{dy}{d\theta} &= -L \cdot \sin\theta \sin\varphi \qquad & \frac{dy}{d\varphi} &= +L \cos\theta \cos\varphi \\
\frac{dz}{d\theta} &= L \cdot \cos\theta \qquad & \frac{dz}{d\varphi} &= 0
\end{aligned}
$$

$$
\begin{aligned}
ds^2 = \quad & L^2 \quad d\theta^2 \\
& 0 \quad d\theta\, d\varphi \\
& L^2 \cos^2\theta \quad d\varphi^2
\end{aligned}
$$

$$\hookrightarrow \quad g_{ij} = \begin{pmatrix} 1 & \\ & \cos^2\theta \end{pmatrix} L^2$$

Notice that it is hard to see from the metric if two manifolds are equal.

Parallel transport

Imagine we would like to
generalize the concept of a "constant field"
in GR. In flat spec the obvious choice would be

$$\partial_\mu S^\lambda(x) = 0$$

which generalizes to

$$\nabla_\mu S^\lambda(x)$$

One attempt to construct such a field
is by following certain paths $x^\mu(\tau)$

$$\frac{D}{D\tau} S^\lambda = \frac{\partial}{\partial\tau} S^\lambda + \Gamma^\lambda{}_{\mu\nu} S^\mu \frac{dx^\nu}{d\tau} = 0$$

This would be called the parallel transport
of S along the path.

Does this depend on the path?

In fact it does, when the space is curved!

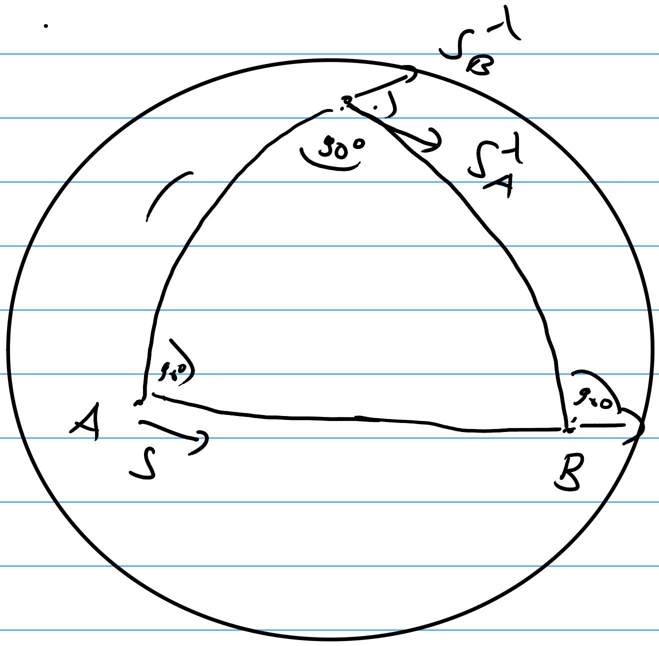
Notice that the scalar product of two parallel transported fields does not change along the path:

$$\frac{D}{D\tau} S^\lambda = 0 \qquad \frac{D}{D\tau} V_\lambda = 0$$

$$\hookrightarrow \quad \frac{D}{D\tau}\left(S^\lambda V_\lambda \right) = \frac{\partial}{\partial\tau}\left(S^\lambda V_\lambda \right) = 0$$

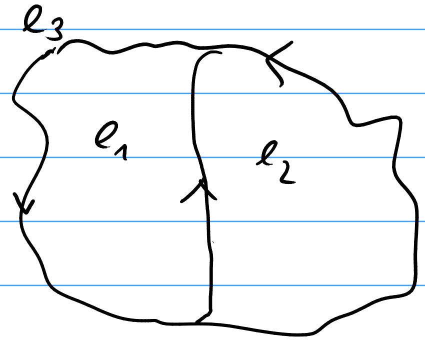
Example: S2 surface. What are the geodesics?

[geodesics are the parallel transported velocities]



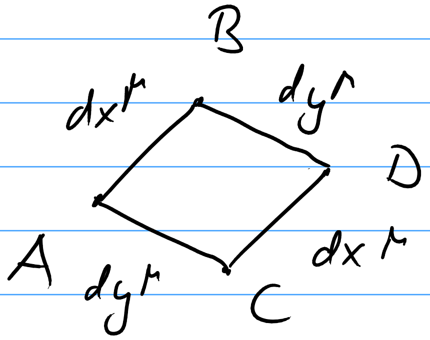
There is no unique way to define a field using parallel transport in a curved space.

Let's mimick the proof of Gauss'/Stoke's theorem: a loop can be split into many smaller loops.



$$\int_{\ell_3} = \int_{\ell_1} + \int_{\ell_2}$$

Consider now parallel transport along two inifinitesimal paths dx^μ and dy^μ



$$S_A^\mu : \quad S_B^\mu = S_A^\mu + dx^\rho\, \Gamma_A{}^\mu{}_{\rho\kappa}\, S_A^\kappa$$

$$S_D^\mu = S_B^\mu + dy^\rho\, \Gamma_B{}^\mu{}_{\rho\kappa}\, S_B^\kappa$$

$$S_D^\mu = S_A^\mu + dx^\rho\, \Gamma_A{}^\mu{}_{\rho\kappa}\, S_A^\kappa$$

$$+ dy^\rho \left(\Gamma_A{}^\mu{}_{\rho\kappa} + dx^\alpha \frac{\partial}{\partial x^\alpha} \Gamma_A{}^\mu{}_{\rho\kappa} \right)$$

$$\times \left(S_A^\kappa + dx^\beta\, \Gamma_A{}^\kappa{}_{\beta\gamma}\, S_A^\gamma \right)$$

(drop A subscripts:)

$$S^\mu_D = S^\mu + dx^\rho \Gamma^\mu_{\rho\kappa} S^\kappa + dy^\rho \Gamma^\mu_{\rho\kappa} S^\kappa$$

$$dy^\rho dx^\alpha \left[\frac{\partial}{\partial x^\alpha} \Gamma^\mu_{\rho\gamma} + \Gamma^\mu_{\rho\kappa} \Gamma^\kappa_{\alpha\gamma} \right] S^\gamma$$

Now consider the difference between A-B-D and A-C-D

$$S^\mu_{ABD} - S^\mu_{ACD} = dy^\rho dx^\alpha S^\gamma$$

$$\times \left[\frac{\partial}{\partial x^\alpha} \Gamma^\mu_{\rho\gamma} - \frac{\partial}{\partial x^\rho} \Gamma^\mu_{\alpha\gamma} + \Gamma^\mu_{\rho\kappa} \Gamma^\kappa_{\alpha\gamma} - \Gamma^\mu_{\alpha\kappa} \Gamma^\kappa_{\rho\gamma} \right]$$

$$\equiv dy^\rho dx^\alpha S^\gamma \; R^\mu{}_{\gamma\rho\alpha}$$

$$R^\mu{}_{\gamma\rho\alpha} \equiv \left[\frac{\partial}{\partial x^\alpha} \Gamma^\mu_{\rho\gamma} - \frac{\partial}{\partial x^\rho} \Gamma^\mu_{\alpha\gamma} + \Gamma^\mu_{\rho\kappa} \Gamma^\kappa_{\alpha\gamma} - \Gamma^\mu_{\alpha\kappa} \Gamma^\kappa_{\rho\gamma} \right]$$

R measures the curvature of the space and parallel transport is path-independent for R=0.

R is called the Riemann tensor.