Resume from last lecture
  evolution equation DGLAP
improvements at small x
  BFKL and non-Sudakovs
  CCFM

http://www-h1.desy.de/~jung/qcdColliderPhysics_2005
Collinear factorisation

- **Generalisation:** applies to any DIS cross section defined by a sum over hadronic final states .... but be careful what it really means....
- **Explicit factorisation theorems exist for:**
  - diffractive DIS (... see above....)
  - Drell Yan (in hadron hadron collisions)
  - single particle inclusive cross sections (fragmentation functions)
- **Practically factorisation works pretty well:**
  - even for jet cross section in pp
  - for heavy quarks in pp
    - BUT watch out multiple interactions (see lecture next year !!!)
    - does not work for diffraction in pp !!!!
  - remains a active area of research !!!!
DGLAP evolution again....

- differential form:
  \[ t \frac{\partial}{\partial t} f(x, t) = \int \frac{dz}{z} \frac{\alpha_s}{2\pi} P_+(z) f \left( \frac{x}{z}, t \right) \]

- differential form using \( f / \Delta_s \) with
  \[
  \Delta_s(t) = \exp \left( - \int_{z_{\text{max}}}^t dz \int_{t_0}^t \frac{\alpha_s}{2\pi} \frac{dt'}{t'} \tilde{P}_2(z) \right)
  \]
  with \( \tilde{P}_2 \sim \frac{1}{1 - z} \)

- integral form
  \[
  f(x, t) = f(x, t_0) \Delta_s(t) + \int \frac{dz}{z} \int \frac{dt'}{t'} \cdot \frac{\Delta_s(t)}{\Delta_s(t')} \tilde{P}(z) f \left( \frac{x}{z}, t' \right)
  \]

  no – branching probability from \( t_0 \) to \( t \)
Sudakov form factor: all loop resummation...

\[ g \rightarrow gg \quad \text{Splitting Fct} \quad \tilde{P}_2(z) = \frac{\tilde{\alpha}_s}{1 - z} \]

- Sudakov form factor .... all loop resummation

\[
\Delta_s = \exp \left( - \int dz \int \frac{dq'}{q'} \frac{\alpha_s}{2\pi} \tilde{P}_2(z) \right)
\]

\[
\Delta_s = 1 + \left( - \int dz \int \frac{dq}{q} \frac{\alpha_s}{2\pi} \tilde{P}_2(z) \right)^1 + \frac{1}{2!} \left( - \int dz \int \frac{dq}{q} \frac{\alpha_s}{2\pi} \tilde{P}_2(z) \right)^2 \ldots
\]

\[
\frac{\tilde{\alpha}_s}{1 - z} \left[ 1 - \int \int dz \frac{dq}{q} \frac{\alpha_s}{2\pi} \tilde{P}_2(z) + \frac{1}{2!} \left( - \int \int dz \frac{dq}{q} \frac{\alpha_s}{2\pi} \tilde{P}_2(z) \right)^2 - \ldots \right]
\]
DGLAP re-sums leading logs...

\[ f(x, t) = f(x, t_0) \Delta_s(t) + \int \frac{dz}{z} \int \frac{dt'}{t'} \cdot \frac{\Delta_s(t)}{\Delta_s(t')} \tilde{P}(z) f \left( \frac{x}{z}, t' \right) \]

- solve integral equation via iteration:

\[ f_0(x, t) = f(x, t_0) \Delta(t) \]

\[ f_1(x, t) = f(x, t_0) \Delta(t) + \int_{t_0}^{t} \frac{dt'}{t'} \frac{\Delta(t)}{\Delta(t')} \int \frac{dz}{z} \tilde{P}(z) f(x/z, t_0) \Delta(t') \]
DGLAP re-sums leading logs...

\[ f(x, t) = f(x, t_0) \Delta_s(t) + \int \frac{dz}{z} \int \frac{dt'}{t'} \cdot \frac{\Delta_s(t)}{\Delta_s(t')} \tilde{P}(z)f\left(\frac{x}{z}, t'\right) \]

solve integral equation via iteration:

\[
\begin{align*}
  f_0(x, t) &= f(x, t_0)\Delta(t) \\
  f_1(x, t) &= f(x, t_0)\Delta(t) + \int_{t_0}^{t} \frac{dt'}{t'} \frac{\Delta(t)}{\Delta(t')} \int \frac{dz}{z} \tilde{P}(z)f(x/z, t_0)\Delta(t') \\
  f_2(x, t) &= f(x, t_0)\Delta(t) + \log \frac{t}{t_0} A \otimes \Delta(t)f(x/z, t_0) + \\
  &\quad \frac{1}{2} \log^2 \frac{t}{t_0} A \otimes A \otimes \Delta(t)f(x/z, t_0) \\
  f(x, t) &= \lim_{n \to \infty} f_n(x, t) = \lim_{n \to \infty} \sum_{n} \frac{1}{n!} \log^n\left(\frac{t}{t_0}\right) A^n \otimes \Delta(t)f(x/z, t_0)
\end{align*}
\]

DGLAP re-sums \( \log t \) to all orders !!!!
for fixed $x$ and $Q^2$ chains with different branchings contribute

iterative procedure, spacelike parton showering

\[ f(x, t) = \sum_{k=1}^{\infty} f_k(x_k, t_k) + f_0(x, t_0) \Delta_s(t) \]
What is happening at small $x$?

- For $x \to 0$ only gluon splitting function matters:

$$P_{gg} = 6 \left( \frac{1-z}{z} + \frac{z}{1-z} + z(1-z) \right) = 6 \left( \frac{1}{z} - 2 + z(1-z) + \frac{1}{1-z} \right)$$

$$P_{gg} \sim 6 \frac{1}{z} \text{ for } z \to 0$$

- Evolution equation is then:

$$f(x, t) = f(x, t_0) \Delta_s(t) + \int \frac{dz}{z} \int \frac{dt'}{t'} \cdot \frac{\Delta_s(t)}{\Delta_s(t')} \frac{\alpha_s}{2\pi} \frac{P(z)}{P(z)} f\left(\frac{x}{z}, t'\right)$$

- At small $z$: $\Delta_s(t) \to 1$

$$xg(x, t) = \frac{3\alpha_s}{\pi} \int_{t_0}^{t} d\log t' \int_{x}^{1} \frac{d\xi}{\xi} \xi g(\xi, t') \quad \text{with} \quad t = \mu^2$$

- When $f(x, t_0)$ is neglected (compared to evolved piece ....)
Estimates at small $x$: DLL

\[ xg(x, t) = \frac{3\alpha_s}{\pi} \int_{t_0}^{t} d\log t' \int_{x}^{1} \frac{d\xi}{\xi} \xi g(\xi, t') \quad \text{with} \quad t = \mu^2 \]

- use constant starting distribution at small $t$:
  \[ xg_0(x) = C \]

\[ xg_1(x, t) = \frac{3\alpha_s}{\pi} \log \frac{t}{t_0} \log \frac{1}{x} C \]

\[ xg_2(x, t) = \left( \frac{3\alpha_s}{\pi} \frac{1}{2} \log \frac{t}{t_0} \log \frac{1}{x} \right)^2 C \]

\[ \vdots \]

\[ xg_n(x, t) = \frac{1}{n!} \frac{1}{n!} \left( \frac{3\alpha_s}{\pi} \log \frac{t}{t_0} \log \frac{1}{x} \right)^n C \]

\[ xg(x, t) = \sum_n \left( \frac{1}{n!} \right)^2 \left( \frac{3\alpha_s}{\pi} \log \frac{t}{t_0} \log \frac{1}{x} \right)^n C \]

\[ xg(x, t) \sim C \exp \left( 2\sqrt{\frac{3\alpha_s}{\pi} \log \frac{t}{t_0} \log \frac{1}{x}} \right) \quad \text{double leading log approximation (DLL)} \]
Results from DLL approximation

- DLL arise from taking small $x$ limit of splitting fct:
  - $\log 1/x$ from small $x$ limit of splitting fct
  - $\log t/t_0$ from $t$ integration...
    gives evolution length....
    softer for running $\alpha_s$
  - strong ordering in $x$ from small $x$ limit
  - strong ordering in $t$ from small $t$ limit of ME...
- DLL gives rapid increase of gluon density from a flat starting distribution
- gluons are coupled to $F_2$... strong rise of $F_2$ at small $x$:

\[ xg(x, t) \sim C \exp \left( 2 \sqrt{\frac{3\alpha_s}{\pi}} \log \frac{t}{t_0} \log \frac{1}{x} \right) \]

consequences:
- rise continues forever ???
- what happens when too high gluon density ?
Applying DGLAP to DIS data ...
Extraction of pdfs from DGLAP fits ...

- Solve DGLAP equations
- adjust input parameters (starting distributions) such that F2 is best described
- extract pdf's as fct of x
- then DGLAP gives pdfs at any Q2
- gluon rises very fast towards small x
  - as predicted als by DLL
Approximations so far ....

- Only inclusive quantities were considered:
  - nothing was said about “real” emissions of gluons or quarks although implicitly assumed....
  - in deriving DGLAP splitting functions we assumed: \( \hat{t} \ll \hat{s} \)

- and also in the small \( t \) limit: \( \hat{t} \sim \frac{-k_{\perp}^2}{1 - z} \)

- neglect \( t \) in previous branchings
  \( t_0 \ll t_1 \ll t_2 \ll t_3 \cdots \ll \mu^2 \)
  - strong ordering condition
  - strong ordering: neglect all kinematics of previous branchings...

- ordering in \( x \)
  \( x_0 > x_1 > x_2 > x_3 \)
Kinematic regions: new evolution ..

- **DGLAP**: strong ordering in $t$
- **DLL**: strong ordering in $t$
  strong ordering in $x$
- what happens if strong $t$ ordering relaxed?

  - **BalitskiiFadinKuraevLipatov** evolution

  - **CataniCiafaloniFioraniMarchesini** evolution
Approximations to higher orders...

gluon bremsstrahlung

\[ \sim \frac{1}{k^2} \left( \frac{1}{z} + \cdots \right) \]

DGLAP
- collinear singularities factorized in pdf
- evolution in \( Q^2 \sim k^2 \), or \( k_t^2 \) or ?

\[ \sigma = \sigma_0 \int \frac{dz}{z} C^a \left( \frac{x}{z} \right) f_a (z, Q^2) \]

BFKL
- \( k_t \) dependent pdf →
- unintegrated pdf
- evolution in \( x \)

\[ \sigma = \int \frac{dz}{z} d^2 k_t \hat{\sigma} \left( \frac{x}{z}, k_t \right) \mathcal{F} (z, k_t) \]
start from integral equation:

\[ f(x, q) = f(x, Q_0) \Delta_s(q) + \int \frac{dz}{z} \int \frac{d^2q'}{\pi q'^2} \cdot \frac{\Delta_s(q)}{\Delta_s(q')} \tilde{P}(z)f\left(\frac{x}{z}, q'\right) \]

use un-integrated pdfs: \[ A(x, k, q) \]

\[ xA(x, k_\perp, q) = xA_0(x, k_\perp) \Delta_s(q) + \int \frac{dz}{z} \int \frac{d^2q'}{\pi q'^2} \cdot \Delta_s(q, f(q')) \tilde{P}(z, q', k_\perp) \Theta(\mathcal{O}) \frac{x}{z} A\left(\frac{x}{z}, k'_l, q'\right) \]

because of phi integration:

\[ \frac{dt}{t} \rightarrow \frac{dq^2}{q^2} \rightarrow \frac{d^2q}{\pi q^2} \]

define updf:

\[ xg(x, Q) = \int \frac{d^2k_\perp}{\pi} xA(x, k_\perp, Q) \Theta(Q - k_\perp) \]

same as before.... but included explicitely dependence on transverse momentum \( k_t \) in addition to evolution scale \( q \)

what are the ordering constraints \( f(q') \) and \( \Theta(\mathcal{O}) \) ?

what is the splitting function?
Approximations: Double Leading Log

- **Recover DLL:**

  - Use

    \[ P_{gg} \rightarrow \frac{3\alpha_s}{\pi} \frac{1}{z} \]

    \[ \Delta_s \rightarrow 1 \]

    \[ \Theta(O) \rightarrow \Theta(k_\perp - k'_\perp) \]

- **Obtain from:**

  \[ xA(x, k_\perp, q) = xA_0(x, k_\perp)\Delta_s(q) + \int dz \int \frac{d^2q'}{\pi q'^2} \cdot \Delta_s(q, f(q')) \tilde{P}(z, q', k_\perp) \Theta(O) \frac{x}{z} \tilde{A} \left( \frac{x}{z}, k'_t, q' \right) \]

  Previous result: In Double Leading Log approximation (upon integration over \( k_t \))

  \[ xg(x, t) = \frac{3\alpha_s}{\pi} \int_{t_0}^{t} d\log t' \int_{x}^{1} \frac{d\xi}{\xi} \xi g(\xi, t') \]
Approximations: BFKL

- At small $z$, divergency of gluon splitting function:
  \[ P_{gg} \sim \frac{1}{z} \]

- Analogy with large $z$ divergency:
  - Cancelled by virtual corrections

- Similar to Sudakov, but NOW at small $x$ .... “non” Sudakov (or Regge form factor)

For $k_\perp \sim k'_\perp$

$q_t \rightarrow 0$

But still $z$ small parallel to $k, k'$
Non-Sudakov form factor: all loop re-sum...

\[ g \to gg \quad \text{Splitting Fct} \quad \tilde{P}(z) = \frac{\alpha_s}{1-z} + \frac{\alpha_s}{z} + \ldots \]

- Non-Sudakov form factor .... all loop resummation

\[
\Delta_{ns} = \exp \left[ -\alpha_s(k_t^2) \int_0^1 \frac{dz'}{z'} \int \frac{dq^2}{q^2} \Theta(k_t - q) \Theta(q - \mu_0) \right]
\]

\[
\Delta_{ns} = 1 + \left( -\alpha_s(k_t^2) \int \frac{dz'}{z'} \int \frac{dq^2}{q^2} \right) + \frac{1}{2!} \left( -\alpha_s(k_t^2) \int \frac{dz'}{z'} \int \frac{dq^2}{q^2} \right)^2 + \ldots
\]

\[
\alpha_s(k_t) \frac{1}{z} \left[ + \alpha_s \log(z) \log \left( \frac{k_t^2}{\mu_0^2} \right) + \frac{1}{2!} \left( \alpha_s \log(z) \log \left( \frac{k_t^2}{\mu_0^2} \right) \right)^2 + \ldots \right]
\]
BFKL equation

- Non-Sudakov form factor screens $1/z$ singularity, 
  ..... as the Sudakov does for $1/(1-z)$

$$\Delta_{ns} = \exp \left( -\bar{\alpha}_s \int \frac{d q^2}{q^2} \int_{z'}^{1} \frac{d z'}{z'} \Theta(k_{\perp}^2 - q^2) \Theta(q^2 - \mu_0^2) \right)$$

$$= \exp \left( \bar{\alpha}_s \log z \log \frac{k_{\perp}^2}{\mu_0^2} \right)$$

$$= z^\omega \text{ with } \omega = \bar{\alpha}_s \log \frac{k_{\perp}^2}{\mu_0^2}$$

$$x A(x, k_{\perp}, q) = x A_0(x, k_{\perp}) + \int \bar{\alpha}_s \frac{dz}{z} z^\omega \int \frac{d^2q'}{\pi q'^2} \frac{x}{z} A \left( \frac{x}{z}, k'_t, q' \right)$$

- here use: $k'_t = k_{\perp} + q$

- recusive equation for BFKL, solve it numerically with iteration...
Different approach to BFKL ...

- use un-integrated pdfs (only kt dependent....) :
  \[ \mathcal{F}(x, Q^2) = x \frac{dg(x, Q^2)}{dQ^2} \]

- relation to: \( \mathcal{F}(x, k_\perp) = x A(x, k_\perp, q) \) at fixed \( q \)
  \[ k'_\perp = k_\perp + q \]

- general form:
  \[ \mathcal{F}(x, Q^2) = \mathcal{F}^{(0)}(x, Q^2) + \int_x^1 \frac{dz}{z} \int dk^2 K(Q^2, k^2) \mathcal{F}(\frac{x}{z}, k^2) \]

- what is \( K(Q^2, k^2) \) ?
\[ \mathcal{F}(x, Q^2) = \mathcal{F}^{(0)}(x, Q^2) + \int_x^1 \frac{dz}{z} \int dk^2 K(Q^2, k^2) \mathcal{F}\left(\frac{x}{z}, k^2\right) \]

- In DGLAP we had:
  \[ K(Q^2, k^2) = \frac{3\alpha_s}{\pi} \frac{1}{Q^2} \Theta(Q^2 - k^2) \text{ for } Q^2 \gg k^2 \]

- Ant-DGLAP (anti-collinear)
  \[ K(Q^2, k^2) = \frac{3\alpha_s}{\pi} \frac{1}{k^2} \Theta(k^2 - Q^2) \text{ for } k^2 \gg Q^2 \]

- What about the region \( Q^2 \sim k^2 \)

- Full BFKL Kernel:
  \[ K(Q^2, k^2) = \frac{3\alpha_s}{\pi} \left( \frac{1}{|Q - k|^2} - \delta(k^2 - Q^2) \int_{k}^{\infty} \frac{d^2q}{\pi q} \right) \]

- Form is fully equivalent with previous form (shown in J. Kwiecinski, A. Martin, P. Sutton PRD 52 (1995) 1445)
relaxing strong ordering of virtualities gives fast increase of gluon at small $x$

*BFKL* gluon increases even faster than with DLL

instead of increasing virtualities... perform a random walk ... increasing or decreasing transverse momentum...

can even reach non-perturbative region ... need cutoff normally set to 1 GeV ...

but result depends on non-perturbative input....
**forward jet production and BFKL**

DIS and forward jet:

- $1.7 < \eta_{jet} < 2.8$
- $x_{jet} > 0.035$
- $0.5 < \frac{p_{t, jet}^2}{Q^2} < 5$
- $\sigma(\text{fwd jet})/\sigma(\text{DIS}) \sim 1\%$

Aim is to investigate and test small $x$ evolution ... Supress contribution from known DGLAP region of phase space
forward jet production

DIS and forward jet:
1.7 < η_{jet} < 2.8
x_{jet} > 0.035
0.5 < \frac{p_{T, jet}^2}{Q^2} < 5

BFKL evolution closer to data
DGLAP/NLO too small
Apply color coherence in form of angular ordering
\[ \bar{q} > \bar{z}_n q_n, q_n > \bar{z}_{n-1} q_{n-1}, \ldots, q_1 > Q_0 \]

with:
\[ \tilde{P}(z, q, k_\perp) = \frac{\bar{\alpha}_s(q(1-z))}{1-z} + \frac{\bar{\alpha}_s(k_t)}{z} \Delta_{NS}(z, q, k_\perp) \]

gives:
\[ x A(x, k_\perp, q) = x A_0(x, k_\perp) \Delta_s(q) + \int dz \int \frac{d^2 q'}{\pi q'^2} \Theta(\bar{q} - zq) \]
\[ \cdot \Delta_s(q, zq') \tilde{P}(z, q', k_\perp) \frac{x}{z} A\left(\frac{x}{z}, k_t', q'\right) \]

integration much more complicated due to angular constraints
Non-Sudakov form factor: all loop re-sum...

\[ g \rightarrow gg \quad \text{Splitting Fct} \quad \tilde{P}(z) = \frac{\bar{\alpha}_s}{1 - z} + \frac{\bar{\alpha}_s}{z} + \ldots \]

- Non-Sudakov form factor .... all loop resummation

\[ \Delta_{ns} = \exp \left[ -\bar{\alpha}_s(k_t^2) \int_0^1 \frac{dz'}{z'} \int \frac{dq^2}{q^2} \Theta(k_t - q) \Theta(q - z'q_t) \right] \]

\[ \Delta_{ns} = 1 + \left( -\bar{\alpha}_s(k_t^2) \int \frac{dz'}{z'} \int \frac{dq^2}{q^2} \right) + \frac{1}{2!} \left( -\bar{\alpha}_s(k_t^2) \int \frac{dz'}{z'} \int \frac{dq^2}{q^2} \right)^2 + \ldots \]

\[ \bar{\alpha}_s(k_t^2) \frac{1}{z} \left[ + \bar{\alpha}_s \log \left( \frac{z}{z_0} \right) \log \left( \frac{k_t^2}{z_0 z q^2} \right) + \frac{1}{2!} \left( \bar{\alpha}_s \log \left( \frac{z}{z_0} \right) \log \left( \frac{k_t^2}{z_0 z q^2} \right) \right)^2 + \ldots \right] \]
Comparison: CCFM and BFKL

Advantage of CCFM:
- attempt to describe emissions
- unified for small and large x

similar behavior
details are different

\[ \lambda = \frac{\partial F}{\partial \log 1/x} \]

H. Jung, QCD & Collider Physics, Lecture 6 WS 05/06

J. Kwiecinski, A. Martin, P. Sutton PRD 52 (1005) 1445