1 Proper Lorentz Transforms

Before we get started let us revise the Lorentz transformation between two equally oriented inertial systems moving with velocity $v$ along the $x^1$-axis. Here $x' = (x'^0, x'^1, x'^2, x'^3)$, $x = (x^0, x^1, x^2, x^3)$ are the space-time coordinates of our inertial systems. With $x^0 := ct$ can write such transformation in coordinates:

\begin{align*}
x'^0 &= \gamma x^0 - \beta \gamma x^1, \\
x'^1 &= -\beta \gamma x^0 + \gamma x^1, \\
x'^2 &= x^2, \\
x'^3 &= x^3,
\end{align*}

where $\beta := v/c$ and $\gamma := 1/\sqrt{1 - \beta^2}$.

Easily, we see that this translates perfectly into matrix operations:

\begin{align*}
\begin{pmatrix}
x'^0 \\
x'^1 \\
x'^2 \\
x'^3
\end{pmatrix} &=
\begin{pmatrix}
\gamma & -\beta \gamma & 0 & 0 \\
-\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x^0 \\
x^1 \\
x^2 \\
x^3
\end{pmatrix}. \number{3}
\end{align*}
We go a step further and define the ’angle’

$$\Psi := \cosh^{-1} \gamma.$$  \hspace{1cm} (4)

Then we have $\gamma = \cosh \Psi$ and $\beta = \tanh \Psi$, which turns our transformation matrix into

$$\begin{pmatrix}
\cosh \Psi & -\sinh \Psi & 0 & 0 \\
-\sinh \Psi & \cosh \Psi & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$  \hspace{1cm} (5)

This reminds us very much of a rotation in $\mathbb{R}^3$ through an angle $\theta$ about the $x^3$-axis:

$$\begin{pmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}.$$  \hspace{1cm} (6)

There are a few differences however. First of all, there is a difference in signs. Moreover, the rotation matrix contains $\sin$, $\cos$, while the matrix of our Lorentz transformation contains $\sinh$ and $\cosh$. Yet, the different signs cause their determinants to be the same. For the rotation matrix it is 1; for our Lorentz transformation matrix it is also $(\cosh \Psi)^2 - (\sinh \Psi)^2 = 1$.

## 2 Four Vectors

In special relativity we work in the 4-dimensional Minkowski space denoted as $\mathbb{M}$, which is an $\mathbb{R}$-vector space with a pseudo scalar product $\langle \cdot, \cdot \rangle_\mathbb{M}$. The latter is different from the Euclidean scalar product. For convenience we will simply write $\langle \cdot, \cdot \rangle$ instead of $\langle \cdot, \cdot \rangle_\mathbb{M}$. Its elements are called *four vectors* $x^\mu = (x^0, x^1, x^2, x^3)$. In physics we choose $x^0 = ct$ with the speed of light $c$ and the spatial coordinates $\vec{x} = (x^1, x^2, x^3)$. While we might be tempted to identify Minkowski space with the 4-dimensional $\mathbb{R}^4$, its structure is inherently different from that of Euclidean space.

**Convention:** In this presentation we are using the *Einstein sum convention*. Over the same indices appearing multiple times, each as a sub- and a superscript, a summation is carried out, e.g. $a_{j,k}^i b_j := \sum_i a_{j,k}^i b_i$.

We can define a *pseudo scalar product* in $\mathbb{M}$:

$$\langle x, y \rangle = x^0 y^0 - (x^1 y^1 + x^2 y^2 + x^3 y^3).$$  \hspace{1cm} (7)

Moreover, we distinct between *covariant* and *contravariant* vectors. If $x^\mu = (x^0, x^1, x^2, x^3)$ is a covariant vector, then $x_\mu = (x^0, -x^1, -x^2, -x^3)$ is its corresponding contravariant vector. The only difference is the different sign in the spatial coordinates. The time coordinate however is left
unchanged. Now we can introduce the *pseudo metric tensor* $g^{\mu\nu}$, $g_{\mu\nu}$ (from now on simply 'metric tensor'). We define its covariant form as

$$g^{\mu\nu} = g_{\mu\nu} := \begin{cases} 1 & , \mu = \nu = 0 \\
-1 & , 1 \leq \mu = \nu \leq 3 \\ 0 & , \mu \neq \nu \end{cases}$$  \hspace{0.5cm} (8)$$

If not otherwise noted $\mu, \nu$ run from 0 to 3. Furthermore, the metric tensor has following important properties:

- $g^{\mu\nu} x_\mu = x_\nu$,
- $g_{\mu\nu} x^\mu = x_\nu$.

So basically, the metric tensor switches indices and turns a covariant into a controvariant, or a controvariant into a covariant vector. We will be using these properties later on.

In fact, we can identify vectors and even scalars with tensors. We call a tensor with $n$ distinct indices a *tensor of $n$-th order*. Thus, a scalar is a $0^{th}$-order tensor, a vector a $1^{st}$-order tensor. The highest order we will deal with in this presentation are $2^{nd}$-order tensors. There are 3 types of them, classified by the position of their indices:

- covariant tensors $\Lambda^{\mu\nu}$,
- controvariant tensors $\Lambda_{\mu\nu}$,
- and mixed tensors $\Lambda_\mu^\nu, \Lambda^\mu_\nu$.

With the aid of this definition we can redefine the scalar product in a more useful manner:

$$\langle x, y \rangle := x^\mu g_{\mu\nu} y_\nu = x^\mu y_\mu.$$  \hspace{0.5cm} (9)$$

If we identify the diagonal matrix $g := \text{diag}(1, -1, -1, -1)$ with the metric tensor, the corresponding vector notation becomes $\langle x, y \rangle = x^T g y$. Now it becomes clear why this is not a real scalar product: The matrix $g$ corresponding to this bilinear form is indefinite.

## 3 Basic Properties of the Transformations

The general homogenous Lorentz transformations are mappings of $\mathbb{M}$ onto itself. They are linear mappings preserving the Minkowski norm $\|x\| := \sqrt{\langle x, x \rangle}$ of four vectors. Like all linear mappings they can be represented by a quadratic matrix $\Lambda$. Thus, the transformation in coordinate form is defined by

$$x'^\mu = \Lambda^\mu_\nu x^\nu.$$  \hspace{0.5cm} (10)$$
The homogenous Lorentz transformations conserve the pseudo norm induced by the scalar product. Let \( x' := \Lambda x \), then

\[
x'^2 = x'^T g x = (\Lambda x)^T g (\Lambda x) = x^T \Lambda^T g \Lambda x = x^T g x = x^2.
\]

Hence, we can conclude that \( \Lambda^T g \Lambda = g \), which gives us

\[
\Lambda^T g \Lambda = g \Rightarrow \det(\Lambda^T \Lambda) = (\det \Lambda)^2 = 1.
\] (15)

So we find \( \det \Lambda = \pm 1 \). This means that \( \Lambda \in \text{Gl}_4(\mathbb{R}) \) is an invertible 4x4-matrix.

Now, let \( \mathcal{L} := \{ \Lambda \in \text{Gl}_4(\mathbb{R}) \mid \Lambda^T g \Lambda = g \} \) be the set of all homogenous Lorentz transformations. Then \( \mathcal{L} \) is a group.

**Proof**

Let \( \Lambda_1, \Lambda_2 \) be in \( \mathcal{L} \). We define \( \Lambda_3 := \Lambda_1 \Lambda_2 \). Then \( \Lambda_3 \) is another homogenous Lorentz transformation since

\[
\Lambda_3^T g \Lambda_3 = \Lambda_2^T \Lambda_1^T g \Lambda_1 \Lambda_2 = \Lambda_2^T g \Lambda_2 = g.
\] (16)

Moreover, the product of any \( \Lambda_i \in \mathcal{L} \) is associative since matrix multiplication is associative. The identity element is the unit matrix \( \mathbb{E}_4 \). Since \( \det \Lambda \neq 0 \) there is an inverse element \( \Lambda^{-1} \in \mathcal{L} \) for each \( \Lambda \in \mathcal{L} \). \( \Box \)

Let us go back to the scalar product in \( \mathbb{M} \). First we contemplate the component with \( \sigma = \rho = 0 \) in \( \Lambda^\mu_0 g_{\mu \nu} \Lambda^\nu_0 = g_{\rho \sigma} \).

We find, that

\[
g_{\mu \nu} \Lambda^\mu_0 \Lambda^\nu_0 = g_{00} (\Lambda^0_0)^2 + \sum_{\mu, \nu=1}^{3} g_{\mu \nu} \Lambda^\mu_0 \Lambda^\nu_0 \] (17)

\[
= g_{00} (\Lambda^0_0)^2 - 3 \sum_{\nu=1}^{3} (\Lambda^0_\nu)^2.
\] (18)

This yields \( 1 \leq (\Lambda^0_0)^2 \iff \Lambda^0_0 \geq 1 \lor \Lambda^0_0 \leq -1 \). Combined with the fact that \( \det \Lambda = \pm 1 \), this implies that \( \mathcal{L} \) is not a connected group. It is made up of four disjoint sets which we call components. Moreover, there are three discrete transformations in \( \mathcal{L} \):

- the identity \( \text{Id} \),
• space inversion $I_S$,
• time inversion $I_T$,
• and space-time inversion $I_{ST} = I_S \circ I_T = I_T \circ I_S = -Id$.

<table>
<thead>
<tr>
<th>symbol</th>
<th>$\Lambda_0^\mu$</th>
<th>$\det \Lambda$</th>
<th>discrete transformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{L}_+^\mu$</td>
<td>$\geq +1$</td>
<td>1</td>
<td>$Id$</td>
</tr>
<tr>
<td>$\mathcal{L}_-^\mu$</td>
<td>$\geq +1$</td>
<td>-1</td>
<td>$I_S$</td>
</tr>
<tr>
<td>$\mathcal{L}_+^\mu$</td>
<td>$\leq -1$</td>
<td>1</td>
<td>$I_T$</td>
</tr>
<tr>
<td>$\mathcal{L}_-^\mu$</td>
<td>$\leq -1$</td>
<td>-1</td>
<td>$-Id$</td>
</tr>
</tbody>
</table>

Table 1: The different components of $\mathcal{L}$ and their properties

The space-time inversion $I_{ST}$ is really just a composition of the first two inversions. The set $\{Id, I_S, I_T I_{ST}\} \subset \mathcal{L}$ is an Abelian subgroup of the Lorentz group. Table 1 gives us an oversight over the four components of $\mathcal{L}$.

Of all the components only the proper orthochronous Lorentz group $\mathcal{L}_+^\mu$ is a subgroup of $\mathcal{L}$. The other components can be identified with left cosets of the corresponding discrete transformation and $\mathcal{L}_+^\mu$. Moreover, the union of $\mathcal{L}_+^\mu$ with any of the other components forms a subgroup of $\mathcal{L}$. From now on, we will restrict our discussion to the proper orthochronous Lorentz group $\mathcal{L}_+^\mu$. Because its parameter space is not bound.

4 Connection to $SL(2, \mathbb{C})$

We want to show the relation of $\mathcal{L}_+^\mu$ to $SL(2, \mathbb{C})$. In doing so, we use the Pauli matrices:

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

With the 2x2-identity matrix $E_2 := \sigma_0$ we define a controvariant 4-tupel

$$\sigma_\mu := (\sigma_0, \sigma_1, \sigma_2, \sigma_3) = (\sigma_0, \vec{\sigma}).$$

It is important to keep in mind that this is now not a 4-tupel of numbers, but a 4-tupel of matrices. We define another such controvariant 4-tupel

$$\vec{\sigma}_\mu := \sigma_\mu = (\sigma_0, -\vec{\sigma}).$$

(19)

With these definitions, we associate to each $x^\mu$ a hermitian 2x2-matrix

$$X = \sigma_\mu x^\mu = \begin{pmatrix} x^0 + x^3 & x^1 - i x^2 \\ x^1 + i x^3 & x^0 - x^3 \end{pmatrix}.$$  

(20)
This means that $X \in \text{Mat}_2(\mathbb{C})$ is self-adjointed, i.e. $\overline{X} = X^T$. First we show that

$$\text{Tr}(\sigma_\mu \sigma_\nu) = 2g_{\mu\nu}. \quad (21)$$

**Proof**

We need following property of the Pauli matrices:

$$\sigma_\mu \sigma_\nu = \delta_{\mu\nu} \sigma_0 + i \cdot \epsilon_{\mu\nu\rho} \sigma_\rho. \quad (22)$$

With the definitions of the controvariant 4-tupels and the properties of the Pauli matrices we now find, that

$$\mu = \nu = 0 \Rightarrow \text{Tr}(\sigma_0 \sigma_0) = 2, \quad (23)$$

$$\mu = 0, 1 \leq \nu \leq 3 \Rightarrow \text{Tr}(\sigma_0 \sigma_\nu) = \text{Tr}(\sigma_\nu) = 0, \quad (24)$$

$$1 \leq \mu \leq 3, \nu = 0 \Rightarrow \text{Tr}(\sigma_\mu \sigma_0) = -\text{Tr}(\sigma_\mu) = 0, \quad (25)$$

$$1 \leq \mu = \nu \leq 3 \Rightarrow \text{Tr}(\sigma_\mu \sigma_\nu) = -\text{Tr}(\delta_{\mu\mu} \sigma_0) = -2, \quad (26)$$

$$1 \leq \mu, \nu \leq 3, \mu \neq \nu \Rightarrow \text{Tr}(\sigma_\mu \sigma_\nu) = -i \cdot \epsilon_{\mu\nu\rho} \cdot \text{Tr}(\sigma_\rho) = 0. \quad (27)$$

Hence, this yields $\text{Tr}(\sigma_\mu \sigma_\nu) = 2g_{\mu\nu}$. Using this, we find the equation

$$\text{Tr}(\sigma^\mu X) = \text{Tr}(\sigma^\mu \sigma_\nu x^\nu) = \text{Tr}(g^{\mu\rho} \sigma_\rho \sigma_\nu x^\nu) = g^{\mu\rho} x^\nu \cdot \text{Tr}(\sigma_\rho \sigma_\nu) = g^{\mu\rho} 2 \cdot g_{\rho\nu} x^\nu = 2g^{\mu\rho} x_\rho = 2x^\mu.$$ 

Moreover, the determinant of $X$ is

$$\det X = \begin{vmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^3 & x^0 - x^3 \end{vmatrix} = (x^0)^2 - (x^3)^2 - ((x^1 - ix^2)(x^1 + ix^2)) = (x^0)^2 - x^2 = x^2. \quad (28)$$

Given a unimodular complex matrix $T \in \{ A \in \text{Gl}_2(\mathbb{C}) | \det A = 1 \}$ we transform the matrix $X$:

$$X' = TXT^\dagger. \quad (29)$$

Clearly, since the determinants of $T, T^\dagger$ equal 1, the Minkowski norm is conserved under such transformation. Hence, $X$ corresponds to a Lorentz transformation $\Lambda$. 

6
5 Generators of $L^\uparrow_+$

Since pure rotations do not affect the $x^0$-coordinate, one can extend the generators of $SO(3)$ by a null row and column for it:

\[
X_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

At last, we have to find the corresponding generator for the Lorentz boost

\[
B_1 = \begin{pmatrix} \cosh \Psi & -\sinh \Psi & 0 & 0 \\ -\sinh \Psi & \cosh \Psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

It is obtained by taking the derivative of $B_1$ at $\Psi = 0$:

\[
\frac{1}{i} \frac{d B_1(\Psi)}{d \Psi} \bigg|_{\Psi=0} = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} =: Y_1.
\]

The $Y_2, Y_3$ for Lorentz boosts along the $x^2, x^3$-axes are found in an analogous fashion. The commutators of these generators are

- $[X_i, X_j] = i \epsilon_{ijk} X_k,$
- $[X_i, Y_j] = i \epsilon_{ijk} Y_k,$
- $[Y_i, Y_j] = -i \epsilon_{ijk} Y_k.$

The first equation shows clearly that the spatial rotations $\{X_1, X_2, X_3\}$ form the subalgebra $SO(3)$. The second equation shows us that the tripl $\{Y_1, Y_2, Y_3\}$ transforms just like a vector under spatial rotations. The last equation shows that the Lorentz boosts $\{Y_1, Y_2, Y_3\}$ do not form a subalgebra, i.e. they do not close in on themselves.

The commutation relations are further simplified by defining the linear combinations

\[
\vec{X}^\pm := \frac{1}{2} \left( \vec{X} \pm i \vec{Y} \right),
\]

where $\vec{X} := (X_1, X_2, X_3)$, and $\vec{Y} := (Y_1, Y_2, Y_3)$. Using this, we acquire the commutations

- $[X_i^+, X_j^+] = i \epsilon_{ijk} X_k^+,$
\[ [X_i^-, X_j^-] = \epsilon_{ijk} X_k^-, \]
\[ [X_i^+, X_j^-] = 0. \]

We see that the first two equations close in on themselves. They form two independent \( SU(2) \) algebras. Further, we note that there is a big similarity to the \( SO(4) \) in this respect. However, these equations here are complex equations. The commutation relations for \( SO(4) \) on the other hand are real equations. Both are \emph{locally isomorphic} to one another.

The irreducible representations \( j_1, j_2 \) of the first and second \( SO(2) \) respectively can now be enumerated. Each has \( 2j_1 + 1 \) degrees of freedom. \( \mathcal{L}_+^1 \) is not compact. Hence, its irreducible representations cannot be unitary. We want to discuss some examples.

- The trivial representation is \((0,0)\).
- The \emph{Weyl} representations \((0, \frac{1}{2}), (\frac{1}{2}, 0)\) are the lowest dimensional irreducible representations with two degrees of freedom.
- The \emph{Dirac} representation \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) is used in quantum electrodynamics to describe the electron. It acts on 4-component spinors.
- The representation \((\frac{1}{2}, \frac{1}{2})\) corresponds to the defining representation of \( \mathcal{L}_+^1 \). Of course, it has four degrees of freedom.
- The representation \((1, 0) \oplus (0, 1)\) with 6 degrees of freedom is carried by antisymmetric tensors \( F_{\mu\nu}. \) In electrodynamics the six independent components can be identified with the electrical and magnetic field components.

\section{Summary}

Let us resume the core aspects of this presentation.

- The Minkowski space \( \mathbb{M} \) is a 4-dimensional \( \mathbb{R} \)-vector space with the pseudo scalar product \( \langle x, y \rangle := x^\mu g_{\mu\nu} y^\nu. \)
- The Lorentz transformations are all endomorphisms of \( \mathbb{M} \) onto itself that leave the Minkowski norm \( \|x\| := \sqrt{\langle x, x \rangle} \) invariant, their absolute determinant is 1.
- The Lorentz group \( \mathcal{L} \) is the set of all such transformations. It is not connected and consists of 4 disjoint sets called components.
- The proper orthochronous Lorentz group \( \mathcal{L}_+^1 \) is the only component that is also a subgroup of \( \mathcal{L} \). Its elements \( \Lambda \) have the properties \( \det \Lambda = +1, \Lambda^0_0 \leq +1 \) and its discrete transformation is \( Id = g^{\mu\nu}. \)
In $SL(2, \mathbb{C})$ we can define a hermitian 2x2-matrix $X := \sigma_\mu x^\mu$, which is invariant under transformation $X' = TXT^\dagger$ with a unimodular matrix $T$. It corresponds to a Lorentz transformation $\Lambda \in L_\uparrow$.

The generators of $L_\uparrow$ are the generators $X_{1,2,3}$ of spatial rotations with an additional null row and column for the $x^0$-coordinate. The generators of the Lorentz boosts $Y_j$ along the $x^j$-axis (j=1,2,3) are obtained by taking the derivative $Y_j := i \frac{dB_j(\Psi)}{d\Psi}|_{\Psi=0}$ at $\Psi = 0$.

The Lie algebra of $L_\uparrow$ is

$$[X^+, X^+] = i\epsilon_{ijk}X^+_k, \quad [X^-, X^-] = i\epsilon_{ijk}X^-_k, \quad [X^+_i, X^-_j] = 0,$$

where $\vec{X}^\pm := \frac{1}{2} \left( \vec{X} \pm i\vec{Y} \right)$.

Literatur