CONSTRAINT'S THEORY AND RELATIVISTIC DYNAMICS

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CONSTRAINT'S THEORY AND RELATIVISTIC DYNAMICS

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Andover research center of Boston University, where the history of quantum gravity was presented in two talks, the first comprising developments up to approximately 1960, and the second, by Prof. A.Ashketar, devoted to recent progress.

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CONTENTS

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Canonical Formalisms
- including Hamilton-Jacobi Theories -
for Classical Fields

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Abstract

By reformulating the variational problem for a given classical Lagrangean field theory in the framework of differential forms, one can show (Lepage) that for $m \geqslant 2$ independent and $n \geqslant 2$ dependent field variables $z^a = f^a(x)$ a much larger variety of Legendre transformations $v^a_\mu = \partial_\mu f^a(x) \to p^\mu_a$, $L \to H$ exists than those which have been employed in physics. Each such theory leads to a Hamilton-Jacobi theory the "wave fronts" of which are transversal to solutions of the field equations. Besides the usual (DeDonder-Weyl) canonical theory which employs the conventional momenta $\pi^\mu_a = \partial L/\partial v^a_\mu$ the canonical theory of Carathéodory is of special interest: its Hamilton function is essentially the determinant of the canonical energy-momentum tensor!

I. Introduction

The fascinating developements of gauge and string theories in recent years have strongly drawn attention to geometrical aspects of field theories, on the classical and the quantum (e.g. anomalies) levels.

There is another very interesting geometrical aspect of field theories, however, which has not yet been exploited by physicists: E. Cartan's geometrical interpretation of partial differential eqs. and its applications by the Belgian mathematician Lepage to "canonical" formalisms for variational Euler-Lagrange field eqs., as a beautiful generalization of the corresponding theory in mechanics where the Hamiltonian framework provides a "canonical formalism" for the Lagrangean eqs. of motion[1]. By a "canonical" formalism I mean the following:

i) Suppose the field eqs. of the system under consideration can be derived as the variational Euler-Lagrange eqs. of an action integral with the Lagrangean density L, then there exists a Legendre transformation which maps the "velocities" v^a_μ $(= \partial_\mu f^a(x)$, $\mu = 0,\ldots,m-1$, $a = 1,\ldots,n$) of the field variables $z^a = f^a(x)$ onto a set of "canonical momentum" variables p^μ_a and which transforms the Lagrangean function $L(x,z,v)$ into a Hamilton function $H(x,z,p)$ the partial derivatives of which generate a set of 1st order partial diff. eqs which replace the 2nd order Euler-Lagrange eqs..

ii) There exists a Hamilton-Jacobi (HJ) part. diff. eqs. generated by the function H, too, the solutions of which describe surfaces ("wave fronts") "transversal" to the m-dimensional extremals. Given certain ("complete") solutions of the HJ eq. one can construct solutions of the Euler-Lagrange eqs. by solving algebraic equations.

iii) The field variables $z^a = f^a(x)$ and the momenta p^μ_a generate a "symplectic" structure. This part of the theory has not been discussed by Lepage and a symplectic structure has been established for special cases only. The existence of a symplectic structure is, of course, of special interest for the problem of quantizing the field theory under consideration.

One important result of Lepage - not yet appreciated by physicists - is that for fields with at least 2 independent and 2 dependent variables there are other qualitatively different canonical theories for a given Lagrangean L than the one conventionally employed in physics. Thus, classical mechanics and the real scalar field cannot be considered the only generic cases for a canonical framework. The present paper summarizes briefly some of the main features of the more general canonical frameworks in question. More details and examples are contained in my recent extensive review[1].

Let me briefly indicate E. Cartan's general idea, how to interprete partial differential eqs. and their solutions $f^a(x)$, $a = 1,\ldots,n$,

in a geometrical way:
The solutions $z^a = f^a(x)$ describe m-dimensional submanifolds Σ^m in $R^{n+m} = \{(x^0,\dots,x^{m-1},z^1,\dots,z^n)\}$. The part. diff. eqs. $\partial_\mu f^a(x) = \varphi^a_\mu(x,z)$ represent conditions on the tangent vectors $X_\mu = \partial_\mu + \partial_\mu f^a(x)\partial_a$, $\mu = 0,\dots,m-1$ of Σ^m. These conditions can be expressed with considerable advantages by giving those "dual" 1-forms ω^a, $a = 1,\dots,n$, which "annihilate" the tangent vectors: $\omega^a = dz^a - \varphi^a_\mu dx^\mu$, $\omega^a(X_\mu) = \partial_\mu f^a(x) - \varphi^a_\mu = 0$. The forms ω^a, $a = 1,\dots,n$, generate an ideal $I[\omega^a]$ in the algebra $\Lambda = \Lambda^0 \oplus \Lambda^1 \oplus \cdots \oplus \Lambda^p \oplus \cdots$ of exterior forms on R^{m+n}.

<u>Criterium of integrability (Frobenius):</u> If $z^a = f^a(x)$ is to be a solution of the part. diff. eqs. $\partial_\mu z^a(x) = \varphi^a_\mu(x,z)$, then the functions $\varphi^a_\mu(x,z)$ have to obey the integrability conditions

$$\partial_\nu\varphi^a_\mu + \varphi^b_\nu\partial_b\varphi^a_\mu = \partial_\mu\varphi^a_\nu + \varphi^b_\mu\partial_b\varphi^a_\nu \ ,$$

or $[X_\mu, X_\nu] = 0$, or $d\omega^a \in I[\omega^a]$.

<u>Rank r of a p-form $\omega^p \in \Lambda^p$:</u> minimal number r of linearly independent 1-forms θ_ρ, $\rho = 1,\dots,r$, by which ω^p can be expressed. One has $r \geq p$. If r = p. then $\omega^p = \theta_1 \wedge \dots \wedge \theta_p$ and ω^p is called decomposable. If $d\theta_\rho \in I[\theta_1,\dots,\theta_r]$, then the r forms θ_ρ define m+n-r completely integrable vector fields on which the θ_ρ vanish and which generate (m+n-r)-dimensional submanifolds in R^{m+n}.

II. Mechanics as an illustration of essential ideas

The introduction of canonical momenta p_j in mechanics can be formulated as a problem of <u>constraints</u>: Let $L(q,v)$ be a Lagrangean function of the 2n variables q^j, v^j, $j = 1,\dots,n$, with the constraint that $v^j = \dot q^j$ on the extremals. As usual the problem can be dealt with by introducing Lagrangean multipliers h_j and a new Lagrangean function $\hat L(q,v,h,q)$ depending on 4n variables:

$$\hat L(q,v,h,\dot q) = L(q,v) - h_j(v^j - \dot q^j) \ . \tag{1}$$

As $\hat L$ does not depend on the time derivative of v^j, the Euler-Lagrange eqs. for these variables are

$$\frac{\partial \hat L}{\partial v^j} = \frac{\partial L}{\partial v^j} - h_j = 0 \ , \quad j = 1,\dots,n \ . \tag{2}$$

Inserting this expression for h_j into $\hat L$, we obtain

$$\hat L = L(q,v) - v^j \frac{\partial L}{\partial v^j} + \dot q^j \frac{\partial L}{\partial v^j} \ .$$

If the eqs. $p_j = \partial L/\partial q^j(q,v)$ can be solved for the variables $v^j = \hat\varphi^j(q,p)$, we can define

$$H(q,p) = \hat\varphi^j(q,p)p_j - L(p, v=\hat\varphi(q,p)) \ , \tag{3}$$

so that

$$\hat L(q,\dot q,p) = -H(q,p) + \dot q^j p_j \ . \tag{4}$$

The 2n Euler-Lagrange eqs. for the variables q^j and p_j,

$$\begin{aligned} \frac{d}{dt}\frac{\partial \hat L}{\partial q^j} - \frac{\partial \hat L}{\partial q^j} &= \dot p_j + \partial_j H = 0, \\ -\frac{\partial \hat L}{\partial p_j} &= \frac{\partial H}{\partial p_j} - \dot p^j = 0, \end{aligned} \tag{5}$$

are the 1st order canonical eqs. of Hamiltonian mechanics! The above considerations can be formulated in a very elegant way in terms of differential forms, a procedure which easily can be generalized to the case of field theories:
The dynamics of the system is determined by the Lagrangean 1-form $\omega = L(q,\dot q)dt$. Let $\omega^j = dq^j - v^j dt$ be the 1-forms which "annihilate" the tangent vectors $e_t = \partial_t - \dot q^j\partial_j$ of the extremals, $\omega^j(e_t) = \dot q^j - v^j = 0$. The 1-forms ω^j generate an ideal $I[\omega^j]$ which vanishes on the extremals. Thus, <u>as far as the extremals are concerned, the form $\omega = Ldt$ is only one representative in an equivalence class of 1-forms, the most general element of which is</u>

$$\Omega = Ldt + h_j\omega^j \ . \tag{6}$$

The coefficient h_j can be determined as follows: We have

$$d\Omega = \left(\frac{\partial L}{\partial v^j} - h_j\right) dv^j \wedge dt + O(\text{mod } I[\omega^j]) . \tag{7}$$

Thus the form Ω is a <u>closed</u> form on the extremals if $h_j = \partial L/\partial v^j =: p_j$. Inserting this expression for h_j into Ω gives

$$\Omega = Ldt + p_j\omega^j = Ldt + p_j(dq^j - v^j dt) = -Hdt + p_j dq^j . \tag{8}$$

We therefore can implement the <u>Legendre transformation</u> $v^j \to p_j$, $L \to H$ by

i) requiring $d\Omega = 0$ (mod $I[\omega^j]$) and

ii) making a change of basis $dt \to dt$, $\omega^j \to dq^j$ in the cotangent space of R^{1+n} and identifying the resulting coefficient of $-dt$ in Ω with the Hamilton function H and the coefficient of dq^j with the canonical momentum p_j!

The associated canonical eqs. of motion are obtained as follows[2]: A 1-parameter variation (homotopy) φ_τ, $\tau \in [-1,+1]$ of the variables q^j, p_j and t, with $\varphi_{\tau=0}$ as the identity mapping, induces a variation of the action integral

$$A = \int_{C(1,2)} \Omega \longrightarrow A_\tau = \int_{\varphi_\tau(C(1,2))} \Omega = \int_{C(1,2)} \varphi_\tau^*(\Omega) , \tag{9}$$

where $C(1,2)$ is a curve $\{(t,q(t),p(t))\}$ over the interval $[t_1,t_2]$ in the $(2n+1)$-dim. extended phase space $\{(t,q,p)\}$. As the extremals $C(1,2)$ are supposed to make the action integral stationary for arbitrary variations φ_τ, provided certain boundary conditions are observed, we have the following necessary condition for the extremals

$$\lim_{\tau\to 0} \frac{1}{\tau}(A_\tau - A_0) = \int_{C(1,2)} \lim_{\tau\to 0} \frac{1}{\tau}[\varphi_\tau^*(\Omega) - \Omega] = \int_{C(1,2)} L(Y)\,\Omega = 0 . \tag{10}$$

Here $L(Y)$ denotes the Lie derivative with respect to the vector field Y which generates the curve $\{(t(\tau),q(\tau),p(\tau)\}=\{\varphi_\tau(t,q,p)\}$. The Lie derivative of an arbitrary exterior form ω can be expressed very conveniently by exterior differentiation d and interior multiplication $i(Y)$ by a vector field Y:

$$L(Y)\omega = i(Y)d\omega + d(i(Y)\omega . \tag{11}$$

($i(Y)$ is defined as follows: $i(Y)$ maps Λ^p into Λ^{p-1}, with $i(Y)f = 0$, $i(Y)df = Yf$, if f is a function, $i(\partial_\mu)dx^\nu = \delta^\nu_\mu$ and $i(Y)(\omega_1\wedge\omega_2) = (i(Y)\omega_1)\wedge\omega_2 + (-1)^p\,\omega_1\wedge(i(Y)\omega_2)$, if ω_2 is a p-form. The eqs. of motion are obtained from the equation

$$\int_{C(1,2)} L(Y)\Omega = \int_{C(1,2)} i(Y)\,d\Omega + i(Y)\Omega\,\Big|_{\partial C(1,2)} = 0 \tag{12}$$

by observing that it should hold for arbitrary vector fields Y with $Y(t_1) = Y(t_2) = 0$. As such a vector can be generated by the special vector fields $\partial_j \equiv \partial/\partial q^j$ and $\partial/\partial p^j$, $j = 1,\dots,n$, we obtain the conditions that the 1-forms

$$\begin{aligned} i(\partial_j)d\Omega &= -(dp_j + \partial_j H dt) =-\theta_j \\ i(\partial/\partial p_j)d\Omega &= dq^j - \frac{\partial H}{\partial p_j}dt = \omega^j \end{aligned} \tag{13}$$

should vanish when applied to the tangent vectors $\partial_t + \dot q^j\partial_j + \dot p_j\partial/\partial p_j$. The result is

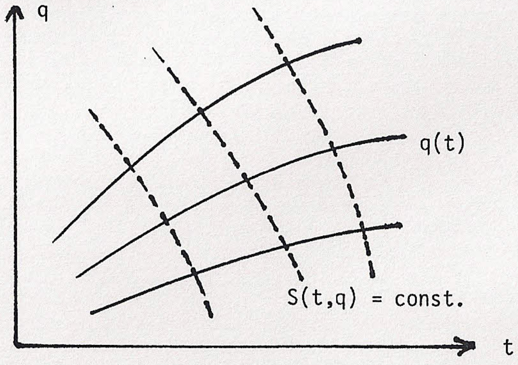
$$\dot q^j = \frac{\partial H}{\partial p_j} , \quad \dot p_j = -\frac{\partial H}{\partial q^j} .$$

The Hamilton-Jacobi eq. can be obtained by the following observation: Since $d\Omega = 0$ on the extramals, it follows (Poincaré's lemma) that locally a function $S(t,q)$ exists such that

$$\Omega = -Hdt + p_j dq^j = dS(t,q) , \tag{14}$$

or by comparing coefficients, $\partial_t S = -H$, $\partial_j S = p_j$, which is the HJ eq.!

The geometrical interpretation of the solutions $S(t,q)$ is the following: Consider a region $G^{1+n} = (t,q)$ in R^{1+n} which is "covered" by extremals, i.e. through each point of G^{1+n} passes exactly one extremal, or, in more modern terms: the extremals $q(t)$ provide a "foliation" of G^{1+n} by 1-dimensional "leaves". In addition, the n-dimensional wave fronts $S(t,q)$ = const. provide a foliation of G^{1+n} by n-dimensional leaves which are <u>transversal</u> to the 1-dim. extremals!



<u>III.</u> <u>Canonical theories for fields in 2 dimensions.</u>

The restrictions to 2 independent variables is convenient, but without any loss of the essential ideas. The dynamics of the system is now determined by a Lagrangean 2-form

$$\omega = L(x,z,v)\, dx^0 \wedge dx^1; \quad x = (x^0,x^1),\ z = (z^1,\dots,z^n)\ , \quad v = (v_0^1,\dots,v_0^n,v_1^1,\dots,v_1^n)\ . \tag{15}$$

On the extremals we have $z^a = f^a(x)$, $v^a_\mu = \partial_\mu f^a(x)$. Again the forms $\omega^a = dz^a - v^a_\mu dx^\mu$ vanish on the tangent vectors $X_\mu = \partial_\mu + \partial_\mu f^a \partial_a$, $\mu = 0,1$, $\partial_a = \partial/\partial z^a$, of the extremals and generate an ideal $I[\omega^a]$. Thus the 2-form ω belongs to an equivalence class of 2-forms, with the general representative

$$\Omega = \omega + \varepsilon_{\mu\nu}\, h^\mu_a\, \omega^a \wedge dx^\nu + \frac{1}{2} h_{ab}\, \omega^a \wedge \omega^b, \tag{16}$$

where $\varepsilon_{01} = 1 = -\varepsilon_{10}$ and the coefficients h^μ_a and $h_{ab} = -h_{ba}$ can be functions of the variables x, z and v. For $d\Omega$ we obtain

$$d\Omega = \left(\frac{\partial L}{\partial v^a_\mu} - h^\mu_a\right) dv^a_\mu \wedge dx^0 \wedge dx^1 + O(\mathrm{mod}\ I[\omega^a]\ . \tag{17}$$

Thus, the form Ω is closed on the extremals if $h^\mu_a = \partial L/\partial v^a_\mu \equiv \pi^\mu_a$. This result is very similar to the corresponding one in mechanics. What is new, however, is that <u>no restriction are imposed on the coefficients</u> h_{ab} without additional requirements! This opens new possibilities for defining canonical frameworks, provided we have $n \geqslant 2$ (for $n = 1$ no $h_{ab} \neq 0$ exists). Defining $d\Sigma_\mu = \varepsilon_{\mu\nu} dx^\nu$ we can write

$$\Omega = \omega + \pi^\mu_a\, \omega^a \wedge d\Sigma_\mu + \frac{1}{2} h_{ab}\, \omega^a \wedge \omega^b\ . \tag{18}$$

Generalizing the procedure discussed for mechanics we can implement a Legendre tranformation by replacing ω^a by $dz^a - v^a_\mu dx^\mu$ and exressing Ω with respect to the basis $dx^0 \wedge dx^1$, $dz^a \wedge dx^\mu$, $dz^a \wedge dz^b$. The result is

$$\Omega = -H\, dx^0 \wedge dx^1 + p^\mu_a\, dz^a \wedge d\Sigma_\mu + \frac{1}{2} h_{ab}\, dz^a \wedge dz^b\ ,$$

$$H = \pi^\mu_a v^a_\mu - \frac{1}{2} h^{\mu\nu}_{ab} v^a_\mu v^b_\nu - L\ , \tag{19}$$

$$p^\mu_a = \pi^\mu_a - h^{\mu\nu}_{ab} v^b_\nu\ , \quad h^{\mu\nu}_{ab} = \varepsilon^{\mu\nu} h_{ab}\ .$$

We see that the Legendre transformation $L \to H$, $v^a_\mu \to p^\mu_a$ depends on the choice of the coefficients h_{ab} (the (invariant) Hamilton function H defined here should not be confused with the energy integral $\int dx^1 T^0{}_0$, where $T^\mu{}_\nu$ is the energy-momentum tensor and which becomes the Schrödinger operator in the corresponding quantum theory). Starting from mechanics ($m = 1$) and the real scalar field ($n = 1$) the usual choice in physics is $h_{ab} = 0$! However, it may be advantagous to have other choices.

Different choices of h_{ab} may be classified according to the rank r of the resulting form Ω: If Ω has rank r, the r 1-forms θ_ρ which generate Ω determine (2+n-r)-dimensional submanifolds. These submanifolds are the Hamilton-Jacobi "wave fronts" associated with the canonical form Ω. For the following discussion it is helpful to define the 1-forms

$$a^\mu = L\,dx^\mu + \pi^\mu_a\,\omega^a = -T^\mu{}_\nu\,dx^\nu + \pi^\mu_a\,dz^a\,,$$
$$T^\mu{}_\nu = \pi^\mu_a\,v^a_\nu - \delta^\mu_\nu\,L\,, \qquad \mu = 0,1\,. \tag{20}$$

Then

$$\Omega = a^\mu \wedge d\Sigma_\mu - L\,dx^0 \wedge dx^1 + \frac{1}{2}\,h_{ab}\,dz^a \wedge dz^b\,.$$

Let me mention the following 2 important examples with ranks 4 and 2, respectively:

i) If $h_{ab} = 0$, then

$$\Omega = \Omega_0 = a^\mu \wedge d\Sigma_\mu - L\,dx^0 \wedge dx^1\,, \tag{21}$$

and we see that Ω_0 has rank 4, because it is generated by the 4 1-forms dx^μ, a^μ.

ii) Probably the <u>most</u> interesting canonical theory is that of <u>Carathéodory</u>: It is characterized by the property that Ω should have the minimal rank 2:

$$\Omega = \Omega_C = \frac{1}{L}\,a^0 \wedge a^1\,, \tag{22}$$

implying

$$H = H_C = -\frac{1}{L}\det\,(T^\mu{}_\nu)\,, \quad h^{\mu\nu}_{ab} = \frac{1}{L}\,(\pi^\mu_a\,\pi^\nu_b - \pi^\mu_b\,\pi^\nu_a)\,.$$

Some properties of the canonical theories associated with the forms Ω_0 and Ω_C are discussed in the next two paragraphs (considerable more material is contained in ref. 1)).

Once a choice of the coefficients h_{ab} has been made, and with it a choice of the momentum functions p^μ_a and the Hamilton function $H(x,p,z)$, the <u>canonical</u> field eqs. are obtained as follows: The 2-forms

$$i(\partial_a)\,d\Omega =: \lambda_a\,, \qquad i(\partial/\partial p^\mu_a)\,d\Omega =: \omega^a_\mu$$

have to vanish when applied to $X_0 \wedge X_1$, where

$$X_\mu = \partial_\mu + \partial_\mu z^a(x)\,\partial_a + \partial_\mu p^\nu_a(x)\,\frac{\partial}{\partial p^\nu_a}$$

is a tangent vector x of the 2-dimensional extremal manifold $\Sigma^2 = \Big\{(x, z(x), p(x)\Big\}$.

<u>Remark:</u> If $d(h_{ab}dz^a \wedge dz^b) = 0$, then the coefficients h_{ab} do not enter the field eqs. and the appearence of the term $h_{ab}\,dz^a \wedge dz^b$ in Ω is equivalent to adding the "total derivative" $(1/2)\,h^{\mu\nu}_{ab}\,v^a_\mu\,v^b_\nu$ to the original Lagrangean L. This happens, for instance, if the h_{ab} are constants.

<u>IV. The canonical theory associated with the (DeDonder-Weyl) form Ω_0.</u>

In this case - the only one usually considered in physics - we have $h_{ab} = 0$ and - as already mentioned in the last paragraph - the basic 2-form is

$$\begin{aligned}\Omega_0 &= a^\mu \wedge d\Sigma_\mu - L\,dx^0 \wedge dx^1 \\ &= -H\,dx^0 \wedge dx^1 + \pi^\mu_a\,dz^a \wedge d\Sigma_\mu\,,\end{aligned} \tag{23}$$

$$a^\mu = L\,dx^\mu + \pi^\mu_a\,\omega^a\,, \qquad H = \pi^\mu_a\,v^a_\mu - L\,, \qquad \pi^\mu_a = \frac{\partial L}{\partial v^a}\,.$$

From

$$\omega^a_\mu = i\left(\frac{\partial}{\partial \pi^\mu}\right)d\Omega_0 = \omega^a \wedge d\Sigma_\mu\,,$$
$$\lambda_a = i(\partial_a)\,d\Omega_0 = -\partial_a H\,dx^0 \wedge dx^1 - d\pi^\mu_a \wedge d\Sigma_\mu\,, \tag{24}$$

one derives the canonical field eqs.

$$v^a_\mu = \frac{\partial H}{\partial \pi^\mu_a}\,, \qquad \partial_\mu \pi^\mu_a = -\frac{\partial H}{\partial z^a}\,, \tag{25}$$

which are equivalent to the Euler-Lagrange eqs..

The HJ theory associated with the canonical form Ω_0 is the following: Since $d\Omega_0 = 0$ (mod $I[\omega^a]$) and since the rank of Ω_0 is 4, we can conclude that locally there are 2 functions $S^\mu(x,z)$, $\mu = 0,1$ such that

$$\begin{aligned}\Omega_0 &= -H\,dx^0 \wedge dx^1 + \pi^\mu_a\,dz^a \wedge d\Sigma_\mu \\ &= dS^\mu \wedge d\Sigma_\mu = dS^0(x,t) \wedge dx^1 + dx^0 \wedge dS^1(x,t)\,.\end{aligned} \tag{26}$$

Comparing coefficients we get the HJ eq.

$$\partial_\mu S^\mu = -H(x,z,\pi)\,, \qquad \pi^\mu_a = \partial_a\,S^\mu(x,z)\,, \tag{27}$$

which - obviously - is a generalization of the HJ eq. $\partial_t S = -H$, $p_j=\partial_j S$ in mechanics.

Let me illustrate several features of the HJ eq. (27) by discussing a special case: $n = 1$, $L = (1/2)(v_0)^2 - (1/2)(v_1)^2 - V(z)$, where $V(z) = (1/2)\mu^2 z^2$ (KG) or $\alpha(1 - \cos(\beta z))$ (SG) or $(1/2)\lambda(z^2 - a^2)^2$.

We have

$$\pi^0 = v_0 , \quad \pi^1 = -v_1 , \quad H = \frac{1}{2}(\pi^0)^2 - \frac{1}{2}(\pi^1)^2 + V(z) ,$$

and therefore the HJ eq. takes the form

$$\partial_\mu S^\mu + \frac{1}{2}(\partial_z S^0)^2 - \frac{1}{2}(\partial_z S^1)^2 + V(z) = 0 . \tag{28}$$

This is <u>one</u> part. diff. eq. for <u>two</u> functions $S^\mu(x,z)$. Thus, one can choose one of them appropriately and then solve the eq. (28) with respect to the other one. This larger freedom reflects somehow the larger freedom of choosing the initial or boundary conditions for the extremals $z = f(x)$ as solutions of a <u>part.</u> diff. eq..

There is, however, the following essential problem which makes it much harder to find interesting solutions of the HJ eq. (27): If $z = f(x)$ is an extremal, then the relations

$$\partial_0 f = \partial_z S^0(x,z) , \quad \partial_1 f = -\partial_z S^1(x,z) \tag{29}$$

can only hold if the integrability condition

$$\partial_1\partial_z S^0 - \partial_z S^1\, \partial_z^2 S^0 = -\partial_0\partial_z S^1 - \partial_z S^0\, \partial_z^2 S^1 \tag{30}$$

is satisfied. Eq. (30) is a part. diff. eq. which has to be solved simultaneously with the HJ eq. (27), a task which does not look easy! A special solution is the following: The seperating ansatz $S^\mu(x,z) = h^\mu(x) + W^\mu(z)$ leads to the solution

$$\begin{gathered} S^0 = -\frac{1}{4} A\, x^0 + \omega\, W(z) , \quad S^1 = -\frac{1}{4} A\, x^1 + k\, W(z) , \\ A, \omega, k = \text{const.}, \quad \omega^2 - k^2 = \mu^2 , \quad W(z) = \frac{1}{\mu}\int dz\, (A - 2V(z))^{1/2} \end{gathered} \tag{31}$$

of the eqs. (27) <u>and</u> (30). Extremals associated with the solutions (31) are: plane waves in the case of the KG-system and 1-soliton solutions in the case of SG- and the z^4-systems.

An important question is, of course, what can you do with a solution $S^\mu(x,z)$, once you found it?

Let me briefly recall some uses of a solution $S(t,q)$ in mechanics:

i) If $q(t)$ is a solution of the 1st order eqs.

$$\dot q^j = \frac{\partial H}{\partial p_j}[q, p_j = \partial_j S(t,q)] = \varphi^j(t,q)$$

then it is extremal.

ii) If $S(t,q;a)$ is a solution of the HJ eq. which depends on a constant a, then

$$G = \frac{\partial S}{\partial a}(t,q(t);a)$$

is a constant of motion for any extremal $q(t)$ for which $p_j = \partial_j S$. Noether's theorem is a special case, where the constant a is a group parameter!

iii) Complete integral: if the solution $S(t,q;a)$ depends on n parameters a_j, such that

$$\left|\left(\frac{\partial^2 S}{\partial q^j \partial a^k}\right)\right| \neq 0 ,$$

then the solutions $q^j(t)$ of the algebraic eqs. $\partial S/\partial a_j = b^j = \text{const.}$ constitute the most general solution of the canonical eqs. of motion.

iv) The solutions $S(t,q)$ are useful for semi-classical (WKB-) approximations of the corresponding quantum system.

Very similar applications hold for solutions of HJ eqs. for fields[1]. Let me mention here just one interesting example: If $S^\mu(x,z;a)$ is a solution of the HJ eq. depending on a constant a, then the current

$$G^\mu(x) = \frac{\partial S^\mu}{\partial a}[x, z=f(x);a] \tag{32}$$

is conserved for any extremal $f^b(x)$ for which the relations $\pi^\mu_b(x) = \partial_b S^\mu[x,z=f(x))]$ hold. This result implies: Let $z^b = f^b(x;a)$ be a solution of the field eqs. which depends on a constant a, then the following current is conserved:

$$G^0(x) = -\pi^0_b \frac{\partial}{\partial a} f^b \,, \tag{33}$$

$$G^1(x) = -\pi^1_b \frac{\partial}{\partial b} f^b + \frac{\partial}{\partial a}\int^{x^1} d\bar{x}^1\, L[f(x^0,\bar{x}^1),\ \partial_\mu f(x^0,\bar{x}^1)] \,.$$

Interesting applications have been discussed by von Rieth[3]: If, e.g., $f^b(x)$ is an instanton-solution of an Euclidean Yang-Mills theory, then one can construct a conserved current for each parameter on which the instanton depends!

In mechanics a symplectic structure can be generated by the 1-form

$$\sigma = p_j\, dq^j = \Omega + H\, dt \tag{34}$$

the exterior derivative $d\sigma = dp_j \wedge dq^j$ of which provides the required symplectic 2-form from which the Poisson brackets may be derived[4]. Similarly[5], for field theories a symplectic structure may be generated by the 2-form

$$\hat{\sigma} = \pi^\mu_a\, dz^a \wedge d\Sigma_\mu = \Omega_0 + H\, dx^0 \wedge dx^1 \tag{35}$$

which, when integrated over the space variable x^1 for x^0 = const. corresponds to the above case in mechanics and provides "equal-time" Poisson brackets for fields! Translated into quantum theory this leads to the canonical quantization of fields in the usual manner.

V. Carathéodory's canonical theory for fields.

As already mentioned in paragraph III this very intereting and beautiful canonical theory is determined uniquely by the requirement that the basic form Ω should have the minimal rank 2. Geometrically this means that the transversal wave fronts have the maximal dimension n! Carathéodory's canonical theory is defined by the following espression for the basic form Ω:

$$\Omega_C = \frac{1}{L}\, a^0 \wedge a^1 = \frac{1}{H_C}\, \theta^0 \wedge \theta^1 \,, \quad \theta^\mu = -H dx^\mu + p^\mu_a\, dz_a \tag{36}$$

which implies

$$H_c = -\frac{1}{L}\, |(T^\mu{}_\nu)| \,, \quad p^\mu_a = -\frac{1}{L}\, \bar{T}^\mu{}_\rho\, \pi^\rho_a \,, \quad T^\mu{}_\nu = \pi^\mu_a v^a_\nu - \delta^\mu_\nu L \tag{37}$$

where $\bar{T}^\mu{}_\nu$ is the algebraic complement of $T^\rho{}_\mu$:

$$T^\mu{}_\rho\, \bar{T}^\rho{}_\nu = \delta^\mu_\nu\, |(T^\mu{}_\nu)| \,, \qquad |(T)| \equiv \det(T) \,.$$

The relations (37) show that Carathéodory's Hamilton function H_C is essentially the determinant of the canonical energy-momentum tensor $(T^\mu{}_\nu)$, an intuitively appealing property. Furthermore, because

$$|(T^\mu{}_\nu = t^\mu{}_\nu - \delta^\mu{}_\nu L)| = L^2 - L\, \mathrm{tr}(t^\mu{}_\nu) + |(t^\mu{}_\nu)| \,, \tag{38}$$

$$t^\mu{}_\nu = \pi^\mu_a v^a_\nu \,,$$

H_C is essentially the characteristic polynomial of the matrix $(t^\mu{}_\nu)$. This means that H_C makes use of <u>all</u> the invariants of this matrix, not just of its trace like $H = \pi^\mu_a v^a_\mu - L$. As

$$H_C = \pi^\mu_a v^a_\mu - L - \frac{1}{L}\, |(t^\mu{}_\nu)| \,, \qquad p^\mu_\nu = \pi^\mu_\nu + O(1/L) \,,$$

we see that the conventional Hamilton function $\pi^\mu_a v^a_\mu - L$ and the associated canonical momenta π^μ_a appear as lowest order approximations if we expand Carathéodory's H_C and p^μ_a in powers of $1/L$!

Carathéodory's Legendre transformation and the canonical field eqs. associated with his theory are quite complicated and I refer again to ref. 1) for more details.

Carathéodory's HJ eq. follows from the relation

$$dS^0(x,z) \wedge dS^1(x,z) = -\frac{1}{H_C}\, \theta^0 \wedge \theta^1 \tag{39}$$

which implies the HJ eq.

$$|(\partial_\mu S^\nu)| + H_C = 0 \,, \qquad p^\rho_a\, \partial_\rho S^\mu = |(\partial_\rho S^\nu)|\, \partial_a S^\mu \,. \tag{40}$$

Carathéodory's theory has the following important structural property: The canonical transformation

$$x^0 \longrightarrow \hat{x}^0 = x^0 \,, \quad x^1 \longrightarrow \hat{x}^1 = S^1(x,z) \,, \quad z^a \longrightarrow \hat{z}^a = z^a \tag{41}$$

casts the theory into Hamiltonian "mechanics" on the surfaces $S^1(x,z) =$ const.. This can be seen from the form Ω_C takes in this frame, namely

$$\Omega_C = \hat{\theta}^0 \wedge d\hat{x}^1 \,, \quad \hat{\theta}^0 = -\hat{H}_C\, dx^0 + p^0_a\, dz^a \,, \quad \hat{H}_C = \hat{T}^0{}_0 \,, \tag{42}$$

and from this form of the canonical field eqs. in the new coordinate system:

$$\begin{gathered} \hat{p}^1_a = 0, \quad \hat{\pi}^1_a = 0, \quad \hat{\pi}^0_a = \hat{p}^0_a, \quad \hat{H}_C = \hat{T}^0{}_0, \\ \frac{d\hat{z}^a}{dx^0} = \frac{\partial \hat{H}_C}{\partial \hat{p}^0_C} \,, \quad \frac{d\hat{p}^0_a}{dx^0} = -\frac{\partial \hat{H}_C}{\partial z^a} \,. \end{gathered} \tag{43}$$

In this coordinate frame one may define a symplectic structure of the type discussed above in paragraph IV.

Carathéodory's canonical theory has a beautiful geometrical structure, but it is complicated algebraically, due to its more complicated Legendre transformation.

A few examples may serve as an illustration[6]:

i) 2 real scalar field with the Lagrangean

$$L = \frac{1}{2} \sum_{a=1}^{2} g^{\mu\nu}\, v^a_\mu\, v^a_\nu \;-\; V(z).$$

Expressing H_C in terms of the canonical variables p^μ_a - see eqs. (19) or (37), gives

$$\begin{gathered} H_C = \frac{1}{2}\,(p_1\cdot p_2 + p_2\cdot p_2) \;+\; V(z)\,\left(1 - \Delta(p)/H_C^2\right) \,, \\ \Delta(p) = (p_1\cdot p_1)(p_2\cdot p_2) - (p_1\cdot p_2)^2 \end{gathered} \tag{44}$$

Thus, H_C is a solution of a cubic equation!

ii) Relativistic string (with the Schild action[7]): From

$$L_S = v^{\alpha\beta}\, v_{\alpha\beta} \,, \quad v^{\alpha\beta} = v^\alpha_1\, v^\beta_2 - v^\alpha_2\, v^\beta_1 \,, \quad a\cdot b = a^0\cdot b^0 - \vec{a}\cdot\vec{b} \,,$$

we get

$$T^\mu{}_\nu = \delta^\mu{}_\nu\, L \,, \quad H_C = -L_S \,, \quad p^\mu_\alpha = -\pi^\mu_\alpha \,,$$

yielding

$$H_C = -\left(\frac{1}{16}\, p_{\alpha\beta}\, p^{\alpha\beta}\right)^{1/3}.$$

iii) Non-linear O(N)-sigma-model[8].

The Lagrangean is

$$L = 2\,\rho^{-2} \sum_{a=1}^{N} (v^a_0)^2 - (v^a_1)^2 \,, \quad \rho = 1 + \sum_{a=1}^{N} (z^a)^2 \tag{45}$$

Despite the rather complicated Legendre transformation $v^a_\mu \to p^\mu_a$ the Hamilton function H_C becomes a simple function of the momenta:

$$H_C = \frac{1}{8}\,\rho^2 \sum_{a=1}^{N} (p^0_a)^2 - (p^1_a)^2 \,.$$

References

1) H.A. Kastrup, Canonical Theories of Lagrangean Dynamical Systems in Physics, Physics Rep. 101 (1983) 1-167. This review contains practically all the relevant literature up to 1983.
2) For more details see chaps. 1 and 2 of ref. 1).
3) J. von Rieth, The Hamilton-Jcobi theory of DeDonder and Weyl applied to some relativistic field theories, J. Math. Phys. 25 (1984) 1102.
4) See, e.g., C. Godbillon, Géometrie Différentielle et Mechanique Analytique, Hermann, Paris 1969, ch. VII; V.I. Arnold, Mathematical Methods of Classical Mechanics, Springer-Verlag Berlin etc. 1978, ch. 8.
5) J. Kijowski and W.M. Tulczyjew, A Symplectic Framework for Field Theories, Lecture Notes in Physics 107, Springer-Verlag, Berlin etc. 1979.
6) More examples and details are contained in ref. 1).
7) A. Schild, Classical Null Strings, Phys. Rev. D16 (1977) 1722. Schild's action is invariant only under reparametrizations with determinant 1. As to theories which are invariant under any non-singular reparametrization see ch. 7 of ref. 1). In addition to the references on string theories given there see: Y. Hosotani, Hamilton-Jacobi Formalism and Wave Equations for Strings, Phys. Rev. Letters 55 (1985) 1719.
8) F. Blum, Institut für Theor. Physik, RWTH Aachen, private communication and to be publ..