Note added in proof. In December 1965 I received from Professor G. Szekeres a reprint of his paper.[19] The inaccessibility of this important reference is doubtless the reason that it has remained unknown to workers in the field. In it, Szekeres independently derives Kruskal's line element. The possibility of identifying opposite events (u,v) and $(-u,-v)$ is briefly mentioned, but rejected on the grounds that it "introduces an artificial singularity at $u=0$, $v=0$, essentially of the same kind as the singularity at the vertex of a cone obtained by identifying the points (x,y) and $(-x,-y)$ of the Euclidean plane." It will be interesting to weigh this objection.

Compare also the recent notes by Belinfante,[20] Anderson and Gautreau,[21] and Rindler.[22]

[19] G. Szekeres, Publ. Math. Debrecen **7**, 285 (1960).

[20] F. J. Belinfante, Phys. Letters (to be published).

[21] J. L. Anderson and R. Gautreau, Phys. Letters (to be published).

[22] W. Rindler, Phys. Rev. Letters **15**, 1001 (1965).

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Position Operators, Gauge Transformations, and the Conformal Group*

H. A. Kastrup†
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey

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The connection between the Lie algebra of the conformal group and the algebra of quantum mechanics is analyzed, applying a method similar to one considered by Segal. The contraction of the 4-parameter special conformal group yields the position operators, and the contraction of the dilatations yields a phase transformation of the states considered. Quantum mechanics appears in this way as a broken symmetry. Thus one gets a relationship between geometrical gauge transformations and phase transformations, which sheds new light on Weyl's conjecture that geometrical gauge transformations and charge conservation are related to each other. Arguments are given as to why the usual interpretation of the special conformal group as a system of transformations connecting frames of constant relative accelerations hardly can be the right one. The main point is that the physically essential group velocity of the wave packets formed by the eigenfunctions of the special conformal group has the same form as in the case of the plane waves, whereas the physically irrelevant phase velocity has the hyperbolic structure usually discussed.

I. INTRODUCTION

In a previous paper,[1] I conjectured that there is a close relationship between the algebra of quantum mechanics and the structure of the conformal group. At that time I was not aware of an earlier interesting paper by Segal,[2] who had already analyzed this question from a more mathematical point of view.

In this paper I wish to discuss the more physical aspects of this problem and shall make some remarks about the relation of the position operators obtained here to those given by Wigner and Newton.[3]

Snyder[4] seems to be the first one who associated the position operators of quantum mechanics with elements of the Lie algebra of a transformation group. As an example of a Lorentz-invariant discrete space-time he considered the position operators given by elements of the Lie algebra of the De Sitter group, which leaves the quadratic form $\eta_0^2-\eta_1^2-\eta_2^2-\eta_3^2-\eta_4^2$ invariant.

In order to allow for continuous translations in the framework of this group, Yang[5] discussed a different set of generators and mentioned as a similar example the Lie algebra of the group which leaves the quadratic form $\eta_0^2-\eta_1^2-\eta_2^2-\eta_3^2-\eta_4^2-\eta_5^2$ invariant.

Segal finally discussed the group with the invariant form $\eta_0^2-\eta_1^2-\eta_2^2-\eta_3^2-\eta_4^2+\eta_5^2$. This group is isomorphic to the conformal group in space-time. He showed by an example how the algebra of quantum mechanics can be obtained as a limit from the Lie algebra of the conformal group.

The following mathematical considerations differ in several aspects from Segal's. We start from a different set of operators and consider only one limiting process instead of two. Furthermore, we give a physical interpretation for this limiting process which seems to be quite natural and which provides a new link between macroscopic physics and atomic quantum mechanics. Its main idea is the following: One of the essential features of low-energy quantum mechanics is its discontinuous energy spectrum, for instance, the spectrum of the hydrogen atom. The conformal group on the other hand implies continuous energies and is, therefore, not compatible with low-energy atomic physics.[6,7] The

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† On leave of absence from the University of Munich.

[1] H. A. Kastrup, Ann. Physik **9**, 388 (1962).

[2] I. E. Segal, Duke Math. J. **18**, 221 (1951).

[3] T. D. Newton and E. P. Wigner, Rev. Mod. Phys. **21**, 400 (1949).

[4] H. S. Snyder, Phys. Rev. **71**, 38 (1947).

[5] C. N. Yang, Phys. Rev. **72**, 874 (1947).

[6] H. A. Kastrup, Nucl. Phys. **58**, 561 (1964).

[7] H. A. Kastrup, Phys. Rev. **142**, 1060 (1966).

transition from conformal physics to quantum mechanics corresponds to a "degeneration" of the algebra of the conformal group into the algebra of quantum mechanics in such a way that the four generators of the special conformal group go over into the position operators and the dilatations go over into a phase transformation of the states considered.

Since such a phase transformation is usually associated with charge conservation, we see how the dilatations can be connected with the conservation of charge. That such a connection might exist was already pointed out in Ref. 6.

This result sheds a new light on Weyl's interesting conjecture that charge conservation is a consequence of geometrical gauge transformations.[8] Despite its fascinating mathematical structure, his hypothesis did not prove very successful physically, and Weyl later even revoked it.[9]

Since the scale transformation is the simplest possible geometric gauge transformation, our results show that there indeed seems to be a close connection between such transformations and the conservation of charge.

Furthermore, we give some arguments as to why the usual interpretation of the special conformal group as a set of transformations which connects systems of constant relative accelerations[10,11] is very unlikely to be the right one. The reason is that the group velocity of the wave packets formed by the eigenfunctions of this group has the same form as that of the plane waves, whereas the phase velocity shows the hyperbolic structure usually related to accelerated motions. Since we know from quantum mechanics that the group velocity, not the phase velocity, describes the motion of particles, we assume this to be true for the conformal group as well. There are several other physical and mathematical reasons why we abandon the interpretation of an accelerated motion and adopt that of a generalized dilatation,[1] until an even better understanding may be found in the future.

We discuss the mathematical problems involved in Sec. II and the physical interpretation in Sec. III. Appendix B contains some critical remarks concerning an example discussed by Fulton *et al.*[11]

II. THE LIE ALGEBRA OF THE CONFORMAL GROUP AND THE ALGEBRA OF QUANTUM MECHANICS

The 15-parameter conformal group in space-time consists of the Poincaré group (inhomogeneous Lorentz group), the dilatations[12]

$$x^{\mu\prime}=\rho x^\mu,\quad \rho>0,\quad \mu=0,1,2,3 \tag{1}$$

and the special conformal transformations

$$x^{\mu\prime}=RT(c)Rx^\mu,\quad \mu=0,1,2,3, \tag{2}$$

where

$$Rx^\mu=-x^\mu/x^2,\quad T(c)x^\mu=x^\mu+c^\mu.$$

If we denote by $M_{\mu\nu}$, P_μ, D, and K_μ the generators of the orthochronous Lorentz group, the translations, the dilatations (1), and the special conformal group (2), respectively, they form the Lie algebra[1]

$$[M_{\kappa\lambda},M_{\mu\nu}]=i(g_{\lambda\mu}M_{\kappa\nu}-g_{\kappa\mu}M_{\lambda\nu}+g_{\kappa\nu}M_{\lambda\mu}-g_{\lambda\nu}M_{\kappa\mu}), \tag{3a}$$

$$[P_\lambda,M_{\mu\nu}]=i(g_{\lambda\mu}P_\nu-g_{\lambda\nu}P_\mu), \tag{3b}$$

$$[K_\lambda,M_{\mu\nu}]=i(g_{\lambda\mu}K_\nu-g_{\lambda\nu}K_\mu), \tag{3c}$$

$$[K_\mu,P_\nu]=2i(g_{\mu\nu}D-M_{\mu\nu}), \tag{3d}$$

$$[P_\mu,P_\nu]=0, \tag{3e}$$

$$[K_\mu,K_\nu]=0, \tag{3f}$$

$$[D,P_\mu]=iP_\mu, \tag{3g}$$

$$[D,K_\mu]=-iK_\mu, \tag{3h}$$

$$[D,M_{\mu\nu}]=0. \tag{3i}$$

In this paper we wish to discuss a special representation of this algebra, namely, the case where the spin S is zero but the squared mass P^2 is not zero. In this section we shall deal mainly with the mathematical problems involved and defer the physical interpretation to the next section.

It was already pointed out in Ref. 7 that all eigenvalues $m^2>0$ occur in such a representation. This is a consequence of the commutation relation[13]

$$e^{i\alpha D}P^2e^{-i\alpha D}=e^{-2\alpha}P^2.$$

It means that an irreducible representation of the conformal group with $P^2\neq0$ is a superposition of irreducible representations of the Poincaré group.[7] In the case of spin zero such a representation is given by the scalar product

$$(\phi_1,\phi_2)=\int_{V_+}d^4p(p^2)^{n-2}\phi_1{}^*(p)\phi_2(p), \tag{4}$$

where V_+ denotes the cone $p^2>0$, $p_0>0$ and n is a real number.

The Hermitian operators of the Lie algebra (3a)–(3i) with respect to the above metric are given by

$$P_\mu=p_\mu, \tag{5a}$$

$$M_{\mu\nu}=i(p_\mu\partial_\nu-p_\nu\partial_\mu), \tag{5b}$$

$$K_\mu=-2n\partial_\mu-2p^\nu\partial_\nu\partial_\mu+p_\mu\Box, \tag{5c}$$

$$D=i(n+p^\mu\partial_\mu), \tag{5d}$$

[8] H. Weyl, *Raum, Zeit, Materie* (Julius Springer-Verlag, Berlin, 1932), Chap. II.

[9] H. Weyl, Naturwiss. **19**, 49 (1931).

[10] An extensive list of literature is given in Ref. 1; see also T. Fulton, R. Rohrlich, and L. Witten, Rev. Mod. Phys. **34**, 442 (1962).

[11] T. Fulton, R. Rohrlich, and L. Witten, Nuovo Cimento **26**, 652 (1962).

[12] We use the units $\hbar=1=c$ and the metric $x^2=(x^0)^2-\mathbf{x}^2$.

[13] J. E. Wess, Nuovo Cimento **18**, 1086 (1960).

with

$$\partial_\mu = \partial/\partial p^\mu, \quad \square = \partial_\mu \partial^\mu.$$

In Ref. 7 we discussed the example $n=1$. In the following we shall choose $n=2$. There is no difficulty in extending the results of this special example to the case of arbitrary $n>0$.

Except for a numerical factor the first part ∂_μ of the operators K_μ represents what one would expect as the form of a position operator in momentum space. Furthermore, the commutation relations (3d) bear a close resemblance to the relations

$$q_j p_k - p_k q_j = i\delta_{jk}, \tag{6a}$$

$$q_j p_k - q_k p_j = M_{jk}, \quad j, k = 1, 2, 3, \tag{6b}$$

in quantum mechanics, we merely have to replace K_μ by $-q_\mu$ and D by $\frac{1}{2}$ to get the relation (6a). In order to describe this situation mathematically we employ the notion of group contractions, first analyzed by Segal[2] and Inonu and Wigner.[14] To this purpose we define the operators

$$P_\mu^{(\epsilon)} = P_\mu, \tag{7a}$$

$$M_{\mu\nu}^{(\epsilon)} = M_{\mu\nu}, \tag{7b}$$

$$K_\mu^{(\epsilon)} = -4\partial_\mu + \epsilon(p_\mu \square - 2p\partial_\nu\partial_\mu), \tag{7c}$$

$$D^{(\epsilon)} = i(2+\epsilon p^\mu \partial_\mu), \tag{7d}$$

where ϵ is a real number with the properties $0 \leqslant \epsilon \leqslant 1$. These operators have the commutation relations

$$[K_\lambda^{(\epsilon)}, M_{\mu\nu}] = i(g_{\lambda\mu}K_\nu^{(\epsilon)} - g_{\lambda\nu}K_\mu^{(\epsilon)}), \tag{8a}$$

$$[K_\mu^{(\epsilon)}, P_\nu] = 2i(g_{\mu\nu}D^{(\epsilon)} - \epsilon M_{\mu\nu}), \tag{8b}$$

$$[D^{(\epsilon)}, P_\mu] = i\epsilon P_\mu, \tag{8c}$$

$$[D^{(\epsilon)}, K_\mu^{(\epsilon)}] = -i\epsilon K_\mu^{(\epsilon)}, \tag{8d}$$

$$[D^{(\epsilon)}, M_{\mu\nu}] = 0, \tag{8e}$$

$$[K_\mu^{(\epsilon)}, K_\nu^{(\epsilon)}] = 0. \tag{8f}$$

The rest of the commutators are the same as before. For $\epsilon=1$ we have the Lie algebra of the conformal group and for $\epsilon=0$ we get the algebra of quantum mechanics $q_\mu p_\nu - p_\nu q_\mu = -ig_{\mu\nu}$, if we define $q_\mu = (1/i)\partial_\mu$. The operators $K_\mu^{(\epsilon)}$ and $D^{(\epsilon)}$ are no longer Hermitian with respect to the scalar product (4) if $\epsilon<1$. For $\epsilon=0$ they are skew Hermitian. This means that their eigenvalues turn from real to imaginary if ϵ goes from one to zero.

We wish to discuss this property in more detail in the case of the operators $K_\mu^{(\epsilon)}$. We call their eigenvalues $l_\mu^{(\epsilon)}$. The eigenfunctions $e^{(\epsilon)}(p)$ are the solutions of the equations

$$[-4\partial_\mu + \epsilon(p_\mu \square - 2p^\nu \partial_\nu \partial_\mu)]e^{(\epsilon)}(p) = l_\mu^{(\epsilon)} e^{(\epsilon)}(p), \quad \mu = 0, 1, 2, 3.$$

The ansatz[7]

$$e^{(\epsilon)}(p) = e(l \cdot p)$$

—we neglect the index ϵ in the following—leads to the ordinary differential equation

$$\epsilon y e'' + 4e' + e = 0,$$

where

$$y = l \cdot p \quad \text{and} \quad e' = de/dy.$$

One can write the solution which is regular for $y=0$ as a series

$$e(y) = 1 - \tfrac{1}{4}y + \frac{1}{2!4(4+\epsilon)}y^2 - \frac{1}{3!4(4+\epsilon)(4+2\epsilon)}y^3 + \cdots,$$

or as[15]

$$e(y) = y^{-\nu/2} J_\nu[2(y/\epsilon)^{1/2}], \quad \nu = (4/\epsilon) - 1,$$

where J_ν is the Bessel function of order ν. For $\epsilon=1$ we have

$$e^{(1)}(y) = y^{-3/2} J_3[2y^{1/2}], \quad l_\mu^{(1)} = h_\mu,$$

and for $\epsilon=0$

$$e^{(0)}(y) = e^{-y/4}, \quad l_\mu^{(0)} = -4ix_\mu.$$

The new feature in the result of our contractions of the conformal group is that we get a position operator for time, too, in contrast to those position operators considered in quantum mechanics and in connection with the representations of the Lorentz group,[3] where one has only space position operators.

The mathematical reason for the possibility of such a position operator for time is the independent integration over p_0 in Eq. (3). This can be seen in the following way: As a consequence of the commutation relations (6a) the operators p_k and q_k have a continuous spectrum.[16] Because of the relation $[q_0, p_0] = -i$, the same applies to the operators p_0 and q_0, at least in the realization discussed above. If these operators are, therefore, considered independent from p_k, q_k, $k=1, 2, 3$, then we need an independent integration over p_0, too, in order to be able to define q_0.

Instead of integrating independently over p_0, we can integrate over $m^2 = p^2$. Since an irreducible representation of the Poincaré group contains only one single mass value,[17] it is clear that such a representation cannot allow for a time position operator in the above sense.

We wish to emphasize, however, that the analogy between p_0, q_0 and p_k, q_k is not complete. The reason is the condition $p_0>0$. As a consequence of this the operator q_0 is not hypermaximal[16] (self-adjoint) with respect to the scalar product (4) but only maximal (symmetric).

We shall discuss the physical implications of the above results in the next section.

In order to obtain the "wave functions" of the

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[14] E. Inonu and E. P. Wigner, Proc. Natl. Acad. Sci. U. S. **39**, 510 (1953); **40**, 119 (1954).

[15] *Higher Transcendental Functions*, edited by A. Erdelyi, (McGraw-Hill Book Company, Inc., New York, 1953), Vol. II. p. 13.

[16] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1955), Chap. II.

[17] E. P. Wigner, Ann. Math. **40**, 149 (1939).

operators q_μ in x space, we have to evaluate the integral

$$\int_{V_+} d^4p\, e^{ip\cdot x}.$$

The details of the calculation are given in Appendix A and the result is

$$\int_{V_+} d^4p\, e^{ip\cdot x} = 4\pi(x^2)^{-2}.$$

This function is singular on the light cone and vanishes for fixed t and large r as r^{-4} and for fixed r and large t as t^{-4}.

Since the conformal group is a simple group of rank 3, one can construct three independent invariant operators[18] which commute with all elements of the Lie algebra of this group and which, therefore, are a multiple of the unity operator in the case of irreducible representations. Something similar holds in the case of the more general algebra (8a)–(8f). For instance, the bilinear invariant is given by

$$J^{(\epsilon)} = \tfrac{1}{2}\epsilon^2 M^{\mu\nu}M_{\mu\nu} + \tfrac{1}{2}\epsilon(P^\mu K_\mu{}^{(\epsilon)} + K_\mu{}^{(\epsilon)}P^\mu) - D^{(\epsilon)2} = 4 - 8\epsilon.$$

Thus we have $J^{(1)} = -4$, $J^{(0)} = 4$. Finally, we want to discuss the finite transformations $e^{iK_\mu c^\mu}$ and $e^{iD\alpha}$. For $\epsilon < 1$, we write $e^{iK_\mu{}^{(\epsilon)}c_\mu{}^{(\epsilon)}}$ and $e^{iD^{(\epsilon)}\alpha(\epsilon)}$.

In order that these operators be unitary for $\epsilon = 0$, the quantities $c_\mu(0)$ and $\alpha(0)$ have to be imaginary. In this way the dilatations degenerate into a phase transformation in the limit $\epsilon = 0$.

III. PHYSICAL INTERPRETATION

We now wish to give a physical interpretation of the mathematical structure described in the last section.

Whereas, the dilatations have a relatively simple structure, the more complicated (and interesting) problems are connected with the special conformal group (2). If we look at the Lie algebra (3a)–(3i) we see that the subalgebra of the Poincaré group and the subalgebra of the homogeneous Lorentz group and the special conformal group are isomorphic. This is a consequence of the definition (2). If we have a unitary representation of one of these subalgebras then we can obtain a unitary representation of the other by the similarity transformation $S(R)$ which represents the transformation by reciprocal radii R. If the unitary representations of the Poincaré group were sufficient to describe the space-time properties of atomic systems, then we might as well use the group which consists of the homogeneous Lorentz group and the special conformal group (2), for these two groups are isomorphic.

The essential difference between the quantities P_μ and K_μ is given by their behavior under dilatations, but before we discuss this point, we wish to analyze the isomorphy of the two groups considered in more detail.

First, let us compare the eigenfunctions of the operators P_μ and K_μ in coordinate space. For the translations they are the plane waves $e^{ip\cdot x}$, and because of Eq. (2), the corresponding functions of the special conformal group are[1]

$$e^{-ih\cdot x/x^2}, \tag{9}$$

where h is a four-vector $(h_0,\mathbf{h})$. Since Bessel-Hagen[19] was the first who considered the quantities h in connection with classical electrodynamics, we shall call them "Bessel-Hagen momenta."

In order to describe the motion of particles by plane waves, we have to build wave packets. In the simplest case such a packet is formed by a superposition of two plane waves with momenta $p_\mu \pm \delta p_\mu$. The space-time position y of the particle is then given by the condition that the two phases of the waves $e^{i(p+\delta p)\cdot x}$ and $e^{ip(x-\delta p)\cdot x}$ are the same:

$$(p+\delta p)\cdot y = (p-\delta p)\cdot y.$$

From this it follows that

$$y_i = \frac{\partial p_0}{\partial p^i} y_0, \quad i=1,2,3,$$

and the group velocity $v_i = \partial p_0/\partial p^i$ is identified with the velocity of the particle.

Exactly the same procedure can be applied to the functions (9), with the result

$$y_i = \frac{\partial h_0}{\partial h^i} y_0, \quad i=1,2,3. \tag{10}$$

Thus we can construct wave packets from the functions (9) which describe motions of particles with a constant velocity. This is in sharp contrast to the behavior of the phases of those functions. The surfaces of a constant phase A are given by

$$h\cdot x = Ax^2. \tag{11}$$

This equation constitutes a quadratic relationship between space and time coordinates. We believe that the difference between the Eqs. (10) and (11) provides the answer to the question why the interpretation of the special conformal group as one which describes systems with constant acceleration[10,11] has not been very successful. This interpretation is connected with the quadratic relation (11). But we know from quantum mechanics that the physically interesting quantity is the group velocity, and that velocity has the same form for the group (2) as for the translations. If one considers the fact that the translations and the group (2) generate the Lorentz group and the dilatations [see Eq. (3d)], it is indeed hard to understand why such accelerations should account for the motions of constant velocity and dilatations. Furthermore, the interpretation of accelerated motions is suggested by taking the nonrela-

[18] Y. Murai, Progr. Theoret. Phys. (Kyoto) 9, 147 (1953).

[19] E. Bessel-Hagen, Math. Ann. 84, 258 (1921).

tivistic limit. But, as we have pointed out several times,[6,7] the extreme relativistic limit seems to be far more interesting for physics.

The difference between Eqs. (10) and (11) corresponds to the difference between group velocity and phase velocity in the case of plane waves.

There is, nevertheless, a difference between the manifolds of the wave packets which can be constructed by plane waves and the functions (9). An expansion of a function $f_1(x)$ in terms of the former is given by

$$f_1(x)=\frac{1}{(2\pi)^4}\int d^4p\,\tilde{f}_1(p)e^{ip\cdot x},$$

and has the inversion

$$\tilde{f}_1(p)=\int d^4x\, f_1(x)e^{-ip\cdot x}.$$

In this case we integrate over the whole p space. The corresponding expansion of a function $f_2(x)$ in terms of the eigenfunctions (9) is

$$f_2(x)=\frac{1}{(2\pi)^4}\int d^4h\,\tilde{f}_2(h)\frac{e^{-ih\cdot x}}{x^2}.$$

It follows immediately from the inversion formula for the Fourier transform of $f_2(Rx)$ that

$$\tilde{f}_2(h)=\int\frac{d^4x}{(x^2)^4}f_2(x)\frac{e^{ih\cdot x}}{x^2}.$$

This means that the "h-transform" of $f_2(x)$ exists if the Fourier transform of $f_2(Rx)$ exists. Thus the functions $f_2(x)$ and $f_1(x)$ form two different classes with a nonvanishing intersection, the properties of which seem to be of particular interest for the theories of quantized fields.[7]

Because of the above considerations we abandon the interpretation of the special conformal group as a system of transformations describing constant accelerations and adopt the interpretation given in Ref. 1. Because of the relation

$$ds^{2\prime}=\sigma^{-2}(x)ds^2,$$
$$ds^2=dx^\mu dx_\mu,\quad \sigma=1-2c\cdot x+c^2x^2,$$

which follows from Eq. (2), we have interpreted that group as a generalized dilatation, the scale factor of which is space and time dependent. There are, of course, a number of geometrical questions connected with this interpretation, too, but we defer them to a later analysis because we have the impression that in the case of the conformal group the more algebraic approach is more promising than the purely geometrical one (see Appendix B).

We have already mentioned that the essential difference between the quantities P_μ and K_μ is their transformation behavior under dilatations: The right-hand sides of Eqs. (3g) and (3h) differ by their sign; the momentum operators have the dimension of length -1, the operators K_μ the dimension $+1$. In other words, because of the relation $RD=-DR$ the transformation of reciprocal radii assures that the eigenvalues of D are symmetric in sign; if s is an eigenvalue of D, then $-s$ is, too. Since the eigenvalues of D describe the dimensions of length of physical quantities,[1] we can characterize the transformation by reciprocal radii R as the "inversion of the dimension of length."

The fundamental mathematical importance of the transformation R can also be seen from the relation (3d). It shows that all the operators $M_{\mu\nu}$ and D can be constructed if the quantities P_μ and K_μ are given. Because of the relation $K_\mu=RP_\mu R$, which follows from Eq. (2), we can generate the whole algebra of the conformal group if the translations and the transformation R are given.

Let us return now to our special example of Sec. II. Since that representation of the conformal group contains a continuous set of masses it cannot describe a single atomic particle in the same sense as the representations of the Poincaré group do. This situation is discussed in detail in Ref. 7 and we can, therefore, confine ourselves here to a few remarks. The dilatations (1) and the generalized dilatations (2) do not allow for the discontinuous energy structure of the atomic world. We, therefore, have to break this symmetry in some way in order to take into account the quantized structure. How this breaking might be done mathematically was described in the last section; we now have to discuss its physical meaning.

We define the Fourier transform $f(x)$ of the functions $\phi(p)$ in coordinate space as

$$f(x)=\int_{V_+}d^4p\;\phi(p)e^{ip\cdot x}.$$

The function $f(x)$ transforms under dilatations as

$$f'(x')=\rho^{-2}f(x)$$

and under the group (2) as $f'(x')=\sigma^2(x)f(x)$. This can be seen easily from the infinitesimal transformations

$$f'(x)=f(x)-\alpha(2+x^\mu\partial_\mu)f(x),\qquad |\alpha|\ll 1,$$
$$f'(x)=f(x)-c^\mu(4x_\mu+2x_\mu x^\nu\partial_\nu-x^2\partial_\mu)f(x),\quad |c_\mu|\ll 1.$$

The operator $D^{(\epsilon)}$ of the last section obviously corresponds to $2+\epsilon x^\nu\partial_\nu$ and the operators $K_\mu^{(\epsilon)}$ to $4x_\mu+\epsilon(2x_\mu x^\nu\partial_\nu-x^2\partial_\mu)$. That part of these operators which is multiplied by ϵ comes from the transformation of the coordinates, the rest of the transformation of the function. In the limit $\epsilon=0$ only the latter remains.

Thus quantum theory is characterized by the fact that the scale or gauge transformations of space-time degenerate into a transformation of the functions without transforming their arguments. In this way the dilatations degenerate into a phase transformation and the special conformal group into position operators. The quantum-mechanical framework usually considered appears if we also drop the time position operator. This "dropping" is associated with a shrinking of the "localized" wave functions; for if we confine ourselves to a single mass value, these functions fall off[3] as e^{-mr} for large distances r, in contrast to the above power law r^{-4}.

Since the phase transformations in quantum mechanics are related to the conservation of charge, one may ask whether the dilatations themselves have something to do with charge conservation, too. This seems indeed to be the case. In an earlier paper[6] we showed by semiclassical arguments that in the case of a very fast particle of charge $\alpha^{1/2}$ in a Coulomb potential generated by a charge $Z\alpha^{1/2}$ the quantity D has the expectation value $Z\alpha$, where α is Sommerfeld's constant and Z an integer.

These results shed some new light on Weyl's conjecture that charge conservation is connected with gauge transformations in space-time.[8] In view of the success of the gauge transformation in quantum mechanics, Weyl revoked his earlier attempts to a large extent.[9] But we have seen that the geometrical picture and the quantum mechanical picture are the two opposite limits of the same structure and that they are not so far apart as Weyl later thought.

In Ref. 7 we mentioned that a representation of the conformal group with a continuous set of mass values can be interpreted in the restricted ("passive"[20]) sense that it describes an isolated atomic particle of a certain mass, the numerical value of which is arbitrary as long as we have not fixed the units of the macroscopic measuring apparatus which determines the mass. In this picture the discarding of the time position operator occurs when the actual measurement of the numerical value of the mass takes place.

All the above considerations show that one has to incorporate the problems connected with the conformal group into the theory of the quantum-mechanical measuring process[21] in order to obtain a theory which deals with all elements involved in such a process. Our deliberations in this paper are merely a first attempt at analyzing this situation.

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[20] A. S. Wightman, Nuovo Cimento, Suppl. **14**, 81 (1959).

[21] See, for instance, Ref. 16, Chaps. V and VI, and for a more recent discussion E. P. Wigner, Am. J. Phys. **31**, 6 (1963).

APPENDIX A

If we introduce the variable $m=(p^2)^{1/2}$ instead of p_0, we have

$$\int_{V_+} d^4p\, e^{ip\cdot x} = \int_0^\infty dm\, m \int_{p_0>0} \frac{d^3p}{2} (\mathbf{p}^2+m^2)^{-1/2} e^{ip\cdot x} = F(x).$$

Except for a constant factor the second integral is the usual $\Delta^{(+)}$ function and its value is[22]

$$\frac{2\pi^2}{r}\frac{\partial}{\partial r} f(r),$$

where

$$\begin{aligned} f(r) &= \tfrac{1}{2}N_0(m\sqrt{(x^2)}) - \tfrac{1}{2}\epsilon(x_0)J_0(m\sqrt{(x^2)}) && \text{for } x^2>0,\\ &= (-1/\pi)K_0(m\sqrt{(-x^2)}) && \text{for } x^2<0; \end{aligned}$$

$$\begin{aligned} \epsilon(x_0) &= 1 && \text{for } x_0>0\\ &= -1 && \text{for } x_0<0. \end{aligned}$$

(J_0 = Bessel function, N_0 = Neumann function, K_0 = Hankel function of the first kind for imaginary arguments; all functions are of order zero.)

For the integration over m we use the following relations[23,24]:

$$\int_0^\infty y^\mu J_0(ay)dy = 2^\mu a^{-\mu-1} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\mu)}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu)}, \qquad -1<\mathrm{Re}\mu<\tfrac{1}{2}, \quad \text{(A1)}$$

$$\int_0^\infty y^\mu N_0(ay)dy = 2^\mu \cot[\tfrac{1}{2}\pi(1-\mu)] a^{-\mu-1} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\mu)}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu)}, \qquad -1<\mu<\tfrac{1}{2}, \quad \text{(A2)}$$

$$\int_0^\infty y^\mu K_0(ay)dy = 2^{\mu-1} a^{-\mu-1} \Gamma^2\left(\frac{1+\mu}{2}\right), \qquad \mathrm{Re}(\mu+1)>0. \quad \text{(A3)}$$

The conditions given in Refs. 23 and 24 for the parameter in Eqs. (A1) and (A2) are not fulfilled in our example. But we can define the left-hand sides by analytic continuation of the right-hand sides as functions of μ, if we avoid the poles of the cotangent and the gamma functions in the numerator. In Eq. (A2) the pole of the cotangent for $\mu=1$ is compensated by the zero of

[22] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience Publishers, Inc., New York, 1959), p. 148.

[23] See Ref. 15, p. 49 and p. 51.

[24] *Tables of Integral Transforms*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1954), Vol. II, p. 97.

$1/\Gamma(\frac{1}{2}-\frac{1}{2}\mu)$. Thus we get

$$\int_0^\infty dm\, mJ_0(m\surd(x^2))=0\,,$$

$$\int_0^\infty dm\, mN_0(m\surd(x^2))=(2/\pi)(x^2)^{-1}\,,$$

$$\int_0^\infty dm\, mK_0(m\surd(-x^2))=-(x^2)^{-1}\,,$$

and therefore

$$\int_0^\infty dm\, mf(r)=(1/\pi)(x^2)^{-1}.$$

After differentiation with respect to r, we finally have

$$F(x)=4\pi(x^2)^{-2}.$$

APPENDIX B

The most detailed discussion of the interpretation of the special conformal group in geometrical terms has been given by Fulton *et al.*[11] These authors analyze different examples and arrive at the conclusion that the group (2) is nothing other than a special case of general relativity, even if one considers it as a gauge transformation in the sense of Weyl. We shall not give a complete clarification here concerning the interpretation of the conformal group as a generalized dilatation and its relationship to general relativity, but we want to point out that the situation is not as simple as the above authors assume. We shall mention two points, the second of which is the essential one.

The most interesting example of Fulton *et al.* is the falling emitter (Sec. 4 of their paper).

In the case of their "conformal" interpretation the authors employ Lorentz' equation of motion and transform the rest mass, according to Schouten and Haantjes,[25] contravariant to the proper time. This is a mathematical possibility, but it is hard to understand from a more physical point of view. As far as we know from quantum mechanics and field theory, the four momenta and, therefore, also the masses are nonlocal quantities, explicitly shown, for instance, by Noether's theorem, applied to translation invariance. It is, therefore, hard to understand if one transforms the mass as a local quantity. In Ref. 7 we gave an example of how the masses are transformed in an irreducible representation of the conformal group without any reference to the coordinate space.

One can illustrate the situation also in the following way: The formal local transformation of a nonlocal quantity has the consequence that no conservation law exists, at least not in the usual sense, as one can see from the Klein-Gordon equation with nonvanishing rest mass.

Yet the main point is the following: The authors start from a frame of reference, where the emitting atom and the emitted light from a system with sharp momenta which, as a consequence of translation invariance, obey the usual energy-momentum conservation law. Then they apply transformations of the group (2), assume again sharp momenta and, invoking translation invariance, the conservation of these momenta, and consequently draw their conclusions.

We believe that these last assumptions are not justified. The reasons can be seen from the commutation relations (3d). Let us consider a linear manifold of states $|p\rangle$ with sharp momenta p_μ which are transformed by the generators K_μ of the special conformal group. It follows from Eq. (3d) that the states $K_\mu|p\rangle$ are not longer eigenstates of the translation operators P_μ. Furthermore, these states do not even form an invariant manifold under the combined groups of translations and special conformal transformations, for these two groups combined do not form a group. To obtain a group, one has to include the Lorentz transformations and the dilatations. Thus the situation with respect to translation invariance of Lorentz' equation is quite different before and after the transformations K_μ. Before these transformations are applied, that equation is invariant under translations and Lorentz transformations separately, but afterwards, it is invariant separately under special conformal and Lorentz transformation, not under translations. The latter invariance can be guaranteed in this last case only by considering the whole 15-parameter conformal group.

The situation becomes a bit clearer if we recall our results of Sec. II: in a conformal invariant theory the momenta p_μ and the Bessel-Hagen momenta h_μ are mathematically on the same footing. This is in sharp contrast to our physical knowledge of them. We shall probably learn more about the physical meaning of the Bessel-Hagen momenta in the future. It may be that their physical importance is small in comparison to that of the momenta, but at least they should not be ignored. The corresponding equation to Lorentz' equation of motion for the momenta is an equation of motion for the Bessel-Hagen momenta, not the former equation in which the mass is transformed like a local quantity.

[25] J. A. Schouten and J. Haantjes, Koninkl. Ned. Akad. Wetenschap. Proc. 39, 1059 (1936).