SOME EXPERIMENTAL CONSEQUENCES OF CONFORMAL INVARIANCE AT EXTREMELY HIGH ENERGIES

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Received 12 November 1962

Apart from the inhomogeneous Lorentz group the conformal transformations in space-time are the following:

The scale transformations

$$x^{\mu'} = \rho x^{\mu} , \quad \rho > 0 \text{ and constant}, \quad \mu = 0,1,2,3, \tag{1}$$

with the property

$$\begin{aligned} ds'^2 &= \rho^2 \, ds^2 , \\ ds^2 &= (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 , \end{aligned} \tag{1a}$$

and the special conformal transformations

$$\begin{aligned} x^{\mu'} &= \sigma^{-1}(x) \, (x^{\mu} - c^{\mu} x^2) , \\ \sigma(x) &= 1 - 2 \, c \cdot x + c^2 \, x^2 , \end{aligned} \tag{2}$$

with the property

$$ds'^2 = \sigma^{-2}(x) \, ds^2 . \tag{2a}$$

The c^{μ}, which form a four vector, are the four independent parameters of the Abelian group (2).

Because of (1a) and (2a) the transformations (1) and (2) may be interpreted as global and local changes of units of length [1]. Here we wish to examine the problem, whether (1) and (2) can be understood also in the sense, that they transform one physical situation into another *. For the scale transformation, for example, this means: If one has a physical object, does there – at least in principle – exist another physical object, which differs from the first one in its extension of length by a factor ρ? In macroscopic physics this certainly is the case. But there are restrictions in microscopic physics. For instance, no particle exists, which differs from the proton only by being 10 times bigger. Similar considerations may be applied to the group (2).

Nevertheless we conjecture, that also in atomic physics the transformations (1) and (2) become important either if the masses of the particles vanish or if their energies are so extremely high, that their masses or any other atomic states of discrete energy are negligible. These conjectures are confirmed by the following considerations:

From (1) and (2) it follows, that the infinitesimal operators of these transformations have the form

$$x^{\nu} \partial_{\nu} , \qquad \partial_{\nu} = \partial / \partial x^{\nu} , \tag{3}$$

$$2 \, x_{\mu} \, x^{\nu} \, \partial_{\nu} - x^2 \, \partial_{\mu} , \qquad \mu = 0,1,2,3. \tag{4}$$

In the case of a free particle we may substitute p_{μ} for ∂_{μ} and therefore expect the following constants of motion, if the theory concerned is invariant under the groups (1) and (2)

$$D = E \, t - \boldsymbol{r}_t \, \boldsymbol{p} \qquad (\hbar = 1 = c) , \tag{5}$$

$$\begin{aligned} K^0 &= 2 \, t \, D - (t^2 - \boldsymbol{r}_t^2) \, E , \\ \boldsymbol{K} &= 2 \, \boldsymbol{r}_t \, D - (t^2 - \boldsymbol{r}_t^2) \, \boldsymbol{p} . \end{aligned} \tag{6}$$

Here $\boldsymbol{r}_t$ is the position-vector of the particle at time t, E its energy and $\boldsymbol{p}$ its momentum. For the time being we further assume that any intrinsic parts are decoupled from the orbit parts (5) and (6). The expressions (5) and (6) may also be derived by using the Lagrange formalism [2].

For a free particle we have $\boldsymbol{r} = \boldsymbol{v} \, t + \boldsymbol{a}$, and thus we obtain instead of (5)

$$D = E \, t - (\boldsymbol{v} \, t + \boldsymbol{a}) \, \boldsymbol{p} = (m^2/E) \, t - \boldsymbol{a} \, \boldsymbol{p} .$$

From this we see, that for $m \to 0$ or $E \to \infty$, D becomes a constant in time. In the same way it may be seen that

$$K^0 \to \boldsymbol{a}^2 \, E$$

$$\boldsymbol{K} \to (\boldsymbol{a}^2) \, \boldsymbol{p} - 2 (\boldsymbol{a} \, \boldsymbol{p}) \, \boldsymbol{a} ,$$

if $m \to 0$ or $E \to \infty$.

Thus, in the limits $m \to 0$ or $E \to \infty$ the quantities (5) and (6) are indeed constants of motion!

The essential question now is, whether there are physically interesting interactions, which are invariant under the groups (1) and (2). There are some, as we shall see. For these interactions we have the following conservation laws under the limiting conditions considered above:

* I am very indebted to some members of the CERN Theory Division, especially to Professor L. Van Hove and to Dr. J. S. Bell, for discussions about this interpretation and critical remarks.

If there are l particles at time t having position-vectors $\boldsymbol{r}_i$, energies E_i and momenta $\boldsymbol{p}_i$, and n particles at time t', corresponding to $\boldsymbol{r}_i'$, E_i' and $\boldsymbol{p}_i'$, then we have in the limits $m_i, m_i' \to 0$ or $E_i, E_i' \to \infty$

$$\sum_{i=1}^{l} t\, E_i - \boldsymbol{r}_i\, \boldsymbol{p}_i = \sum_{i=1}^{n} t'\, E_i' - \boldsymbol{r}_i'\, \boldsymbol{p}_i' \,,$$

$$\begin{aligned} &\sum_{i=1}^{l} 2t(t\, E_i - \boldsymbol{r}_i\, \boldsymbol{p}_i) - (t^2 - \boldsymbol{r}_i^2)\, E_i \\ &\quad = \sum_{i=1}^{n} 2t'(t'\, E_i' - \boldsymbol{r}_i'\, \boldsymbol{p}_i') - (t'^2 - \boldsymbol{r}_i'^2)\, E_i' \,, \\ &\sum_{i=1}^{l} 2\boldsymbol{r}_i(t\, E_i - \boldsymbol{r}_i\, \boldsymbol{p}_i) - (t^2 - \boldsymbol{r}_i^2)\, \boldsymbol{p}_i \\ &\quad = \sum_{i=1}^{n} 2\boldsymbol{r}_i'(t'\, E_i' - \boldsymbol{r}_i'\, \boldsymbol{p}_i') - (t'^2 - \boldsymbol{r}_i'^2)\, \boldsymbol{p}_i' \,. \end{aligned} \tag{8}$$

The relations (7) and (8) may be useful at extremely high energies for the kinematical analysis of experiments corresponding to conformal invariant interactions, for instance for electron-electron colliding beam experiments, future accelerators for protons etc. of some hundred GeV and in cosmic ray physics.

A simple application of (7) is the following: Let us consider the elastic scattering of two identical particles in the center of mass system at extremely high energies — for instance, the elastic scattering of two electrons in a colliding beam experiment. We here ignore the spins and all other intrinsic properties of the particles.

The essential point is now, that the quantity D is the phase of the particle wave at the centre of the wave packet, which in the particle picture corresponds to the position of the particle. Since (7) connects these phases before and after the scattering, we can derive a relation for the phase shift η at extremely high energies:

In the c.m. system we have $\boldsymbol{p}_1 + \boldsymbol{p}_2 = 0$. Therefore we put $\boldsymbol{p}_1 = \boldsymbol{p}$, $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ on the left hand side (7) with corresponding definitions on the right hand side. If we have $\boldsymbol{r} = \boldsymbol{v}\, t$ before the scattering, then the left hand side of (7) vanishes in the limit $E_1, E_2 \to \infty$. After the scattering we have because of the phase shift η [3]

$$r' = v'\, t' - 2(\mathrm{d}\eta/\mathrm{d}p') \,. \tag{9}$$

If we insert this into the right hand side of (7), then, by the same arguments as above, we have the important relation

$$p'(\mathrm{d}\eta/\mathrm{d}p') \to 0 \quad \text{for} \quad p' \to \infty \,. \tag{10}$$

Since $p' = p$, this means

$$\eta \sim p^{-\epsilon} \,, \quad \epsilon > 0 \,, \quad \text{for} \quad p \to \infty \,.$$

Eq. (10) holds for all partial waves. For small η we have $\exp(2i\eta) - 1 \approx 2i\eta$. Thus, if we make the simplifying assumption, that ϵ is the same for all partial waves, we obtain for the asymptotic energy dependence of the elastic cross section σ_{el} in the case of conformal invariant interactions

$$\sigma_{\mathrm{el}} \sim E^{-2(1+\epsilon)} \,, \quad \epsilon > 0 \,, \quad \text{for} \quad E \to \infty \,, \tag{11}$$

with $E = E_1 = E_2 \sim P$.

Since at very high energies the inelastic reactions dominate, our considerations are certainly somewhat too simple, but they show that the relations (7) and (8) imply immediate physical consequences, which may be checked by experiment.

Among others the following interactions $L(x)$ are invariant under the transformations (1) and (2)

$$\bar{\psi}\, \gamma^\mu\, \psi A_\mu \,, \quad \bar{\psi}\, \gamma^5\, \psi\, \varphi \,, \quad \varphi^4 \,, \tag{12}$$

whereas, for example, the interactions

$$\bar{\psi}_1\, O_{\mathrm{i}}\, \psi_2\, \bar{\psi}_3\, O_{\mathrm{i}}\, \psi_4 \,, \quad \bar{\psi}\, \gamma^\mu\, \gamma^5\, \psi\, \partial_\mu\, \varphi \tag{13}$$

are not invariant (O_{i} means any element of the Dirac algebra). Invariance here means, that the quantity $L(x)\mathrm{d}^4x$ is invariant, which is sufficient in the Lagrangian formalism.

The infinitesimal transformations of the (Lorentz-) scalar $\varphi(x)$, the vector $A_\mu(x)$ and the spinor $\psi(x)$ are the following ($\delta u = u'(x') - u(x)$)

$$\begin{aligned} \delta\varphi &= -\,\alpha\, \varphi(x) \,, \quad |\alpha| \ll 1 \,; \quad \rho = \mathrm{e}^{\alpha} \,, \\ \delta\varphi &= -\,2\, c\cdot x\, \varphi(x) \,, \quad |c^\mu| \ll 1 \;\; ^{5)}; \end{aligned} \tag{14}$$

$$\begin{aligned} \delta A_\mu &= -\,\alpha\, A_\mu(x) \,, \\ \delta A_\mu &= -\,2\, c\cdot x\, A_\mu(x) - 2(x^\nu\, c_\mu - x_\mu\, c^\nu)\, A_\nu(x) \;\; ^{3)}; \end{aligned} \tag{15}$$

$$\begin{aligned} \delta\psi &= -\tfrac{3}{2}\,\alpha\, \psi(x) \,, \\ \delta\psi &= (-2\, c\cdot x - c_\mu\, x_\nu\, \gamma^\mu\, \gamma^\nu)\, \psi(x) \;\; ^{5)} \,, \end{aligned} \tag{16}$$

where the γ^μ are Dirac matrices.

The prescription for finding these transformation laws follows from the interpretation of (1) and (2): First introduce the units of length, velocity and action! If then any spinor or tensor A has the dimensions of (length)^n, it has to be transformed under (1) as $A \to \rho^n A$. As for (2), build a Lorentz invariant I of A and if this invariant has the dimension of (length)^m it has to be transformed as $I \to \sigma^{-m}(x)$. For instance, the dimension of length of $\bar{\psi}\psi$ is (L^{-3}). Therefore $\bar{\psi}\psi \to \sigma^3\bar{\psi}\psi$.

From

$$\begin{aligned} \sigma(x) &= 1 - 2\, c\cdot x + c^2\, x^2 \\ &= (1 - c_\varkappa\, x_\lambda\, \gamma^\lambda\, \gamma^\varkappa)\,(1 - c_\mu\, x_\nu\, \gamma^\mu\, \gamma^\nu) \end{aligned}$$

if then follows that

$$\psi'(x') = \sigma(x)\,(1 - c_\mu\, x_\nu\, \gamma^\mu\, \gamma^\nu)\, \psi(x) \,,$$

which for $|c^\mu| \ll 1$ is identical with the second line in (16).

Since in ordinary quantum field theory the limit $E \to \infty$ always leads to troublesome divergences, conformal invariance may also be of theoretical interest in this connection, particularly because the mathematical framework of the inhomogeneous Lorentz group has to be extended essentially, if one deals with the conformal group [1].

I am grateful to Professor F. Bopp for stimulating discussions and his kind interest in this work. I also thank M. Rinke for discussions and a critical reading of the manuscript.

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ON THE RELATION BETWEEN LANDAU DAMPING AND ENERGY ABSORPTION

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Received 2 November 1962

In a recent paper [1] Ching-Sheng Wu has pointed out and corrected some errors contained in a paper by the present writer [2].

As the latter author himself in other papers has done the calculations concerned correctly [3,4] based, however, on another physical point of view than that advocated by Ching-Sheng Wu, some clarifying comments seem worth while in order to highlight the underlying ideas.

In the paper under discussion the Landau damping was calculated from the point of view of energy absorption of resonant particles and the correct expression for the damping was found except for a factor $\frac{1}{2}$ *.

To calculate the absorbed energy A_{res} the integral

$$A = -e\int_{-\infty}^{\infty} \overline{E\, v\, f_1(x,v,t)}\,\mathrm{d}v \qquad (1)$$

was used.

The notation is: $E = E_0' \exp \mathrm{i}(kx - \omega' t)$, the electric wave field, k = wave number, $\omega' = \omega - \mathrm{i}\gamma$ is the complex frequency, f_1 is the perturbation of the distribution function $f_0(v)$, v = velocity of the electrons along the x-direction and $-e$ is the charge of an electron. The bar indicates mean value in space, i.e., over one wavelength.

The electric wave field E is assumed unknown a priori and the damping γ is introduced as an auxiliary quantity, being useful when applying a normal mode analysis to the present problem.

When E is inserted into the Vlasov equation, an expression forth e perturbation f_1 is derived. The real part of the latter is then introduced into (1) together with the real part of E. A is then calculated in the limit $\gamma \to 0$ and an expression for A independent of γ is found †. Or, A is derived to zeroth order in γ.

This means that the derived absorption is solely due to resonance and is the contribution to A in which we are interested. From a physical point of view it is also reasonable that the resonance absorption should be independent of γ in the limit $\gamma/\omega \to 0$. In that case we can write $A = A_{\mathrm{res}}$.

As a consequence of this non-zero absorption there cannot exist in the plasma an intrinsic neutral wave, since in this case the resonant energy must be taken from the wave itself.

Now, what is meant by taking energy from the wave? Two points of view may be set forth, and here lies the difference (or synthesis) of the two ways of correcting the error already mentioned, the lack of $\frac{1}{2}$. One can argue as follows.

a. Due to the resonant energy absorption the averaged electric field energy density $W_{\mathrm{el}} = \overline{E^2}/\gamma\pi$ is damped and thus also the electric field E. This damping leads in turn to a damping of the perturbation speed of all the particles of the main plasma. Consequently the averaged kinetic energy density W_{kin} of the main plasma, as distinguished

* What concerns the error in sign, there is agreement between C.S.Wu and the present author [3,4].

† A may depend on , but only through E.