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GAUGE PROPERTIES OF THE GALILEI SPACE

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Abstract: It is shown that the free non-relativistic motion is invariant not only under the 10-parameter Galilei group but also under the larger 15-parameter Liouville group, which is isomorphic to the group O(2,4). In addition, there is the usual invariance under time translations and special Galilei transformations. The conserved quantities associated with the new symmetries are given explicitly. Further, it is shown in the case of spin zero that there is a close connection between the unitary representations of the Louiville group and the physical projective representations of the Galilei group.

Finally, the consequences of the new conservation laws for interactions, in particular for elastic scattering, are discussed, and it is shown that they impose a vanishing time delay during the interaction. This means that the new exact invariances of the free non-relativistic particles in general can only be approximate or limiting symmetries for interacting systems.

I. INTRODUCTION

The 10-parameter Galilei group $\mathcal{G}_{10}$ is generally considered to be the non-relativistic analogue of the relativistic Poincaré group $\mathcal{P}_{10}$. The proper Galilei group consists of the space translations $T_3(\boldsymbol{a})$, the time translations $T_1(\tau)$, the rotations $R_3(\omega)$ and the special Galilei transformations $G_3(\boldsymbol{b})$. The indices are to indicate the number of independent parameters. The proper Galilei group induces the following well-known infinitesimal transformations of the space and time coordinates $\boldsymbol{x}$ and t:

$$T_3(\boldsymbol{a}): \ \boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{a}, \qquad t \to t, \tag{1a}$$

$$T_1(\tau): \ \boldsymbol{x} \to \boldsymbol{x}, \qquad t \to t + \tau, \tag{1b}$$

$$R_3(\omega): \ x^i \to x^i + \omega^{ik} x^k, \quad \omega^{ik} = -\omega^{ki}, \quad i,k = 1,2,3, \tag{1c}$$

$$G_3(\boldsymbol{b}): \ \boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{b}t, \qquad t \to t. \tag{1d}$$

Every closed non-relativistic system is expected to be invariant under these transformations. The invariance yields the usual ten conservation laws for closed systems.

The situation seems to be rather uncomplicated as far as the classical non-relativistic systems are concerned. In a naive approach one would expect the unitary faithful representations of the Galilei group to provide the appropriate spaces for quantum mechanical physical systems. However, this has been shown [1] not to be the case. Rather, only the faithful representations of a certain central extension of the Galilei group are the physically interesting ones [2, 3].

This is a somewhat unusual situation, although several arguments can be given in order to explain it. The present paper gives - to some extent - an additional explanation for these unconventional properties of the Galilei group in quantum mechanics. The starting point for the following approach has been this:

If one asks for the geometrical gauge transformations of the Minkowski space, namely those transformations of the coordinates x^i, $i = 0,1,2,3$, which induce a multiplication of the line element $ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$ by a factor, one ends up [4] with - besides the full Poincaré group - the dilatations

$$D_1(\alpha): \quad x^i \to e^\alpha x^i, \quad i = 0,1,2,3, \tag{2a}$$

the special conformal transformations

$$SC_4(c): \quad x^i \to RT_4(c)Rx^i, \quad i = 0,1,2,3, \tag{2b}$$

where

$$T_4(c)x^i = x^i + c^i, \quad Rx^i = -x^i/x^2$$

and in particular the discrete length inversion R which in combination with the translations generates the group $SC_4(c)$.

In the future I shall call the group SC_4 the "special Liouville group of the Minkowski space", because Liouville was the first one [5] to show the conformal transformations of the three-dimensional space to form a group with a *finite* number of parameters, contrary to two dimensions where all holomorphic functions provide conformal mappings. In addition I shall call the full 15-parameter conformal group, consisting of the full Poincaré group, the dilatations and the special Liouville group [6] the "full Liouville group $\mathcal{C}_{15}$ of the Minkowski space".

For many years the spacial part of the group (2b), characterized by the parameters (c^1, c^2, c^3), has been interpreted [7] as a transformation of a physical system at rest to a uniformly accelerated system ("hyperbolic motion"). This has been criticized by the author [4, 8, 9] for several reasons:

(i) In eqs. (2b) the transformations R and SC_4 are not well defined in x-space, because, for instance, the light cone $x^2 = 0$ has no well-defined image with respect to R. It is therefore not conceivable that these transformations can be given a well-defined meaning in x-space.

A one-to-one mapping can be obtained by introducing the homogeneous coordinates $x^i = \eta^i/\kappa$, where η^i denotes the *position* of a given physical point in space-time and κ characterizes the inverse Poincaré invariant *unit of length* employed at this point. On this new manifold, which has a simple and natural physical interpretation [4], the length inversion R leaves the position of a point invariant but changes the quantity κ by a different factor at different points. Thus, the transformation R is obviously a geometrical gauge transformation. Since the special Liouville group is composed of two discrete gauge transformations and a translation, it is hard to understand why it should describe hyperbolic motions.

(ii) In any continuous unitary representation of the proper orthochronous Liouville group in which the integrated commutation relations

$$e^{i\alpha D}\, P_0\, e^{-i\alpha D} = e^{-\alpha}\, P_0\,, \quad e^{i\alpha D} P^2\, e^{-i\alpha D} = e^{-2\alpha} P^2$$

between the self-adjoint generator D of the dilatations, the energy operator P_0 and the mass operator P^2 hold, the spectra of P_0 and P^2 are either continuous or vanish [10]. The crucial point now is this: invariance under translations and the special Liouville group implies invariance under dilatations [4, 11]. In the case of non-vanishing rest masses this means that uniformly accelerated systems would have continuous rest masses and continuous energy spectra. However, there is no experimental evidence that the discrete energy spectra of atoms or the rest masses of elementary particles become continuous under uniform accelerations.

On the other hand, gauge transformations of different types have been very useful as approximate symmetries in the very high energy region when rest masses become negligible [4, 9, 12]. The interpretation of the special Liouville group as a geometrical position dependent gauge transformation does not run into so many epistomenological difficulties as the "acceleration" interpretation does.

(iii) The wave packets formed by superpositions of eigenfunctions of the generators K^j of the infinitesimal special Liouville transformations describe certain motions in space-time, analogous to the usual wave packets formed by plane waves. The *group velocity* of these new wave packets is a constant [8], smaller than the velocity of light or equal to it, whereas the *phase velocity* can be larger than the velocity of light and describes hyperbolic motions. Since we know from quantum mechanics that the group velocity, not the phase velocity, of wave packets corresponds to the motion of particles, the "acceleration"-interpretation is again in trouble, but the "gauge"-interpretation is not!

Now, if it is true that - according to our "gauge"-interpretation - the mass "gap" in the relation $E = c(\boldsymbol{p}^2 + c^2m^2)^{\frac{1}{2}}$ is the reason why the relativistic Liouville group is only an approximate symmetry group and that any similarity to uniform accelerations is irrelevant and accidental, then the Liouville group should become important for any free elementary excitation without an energy gap in its dispersion law $E = E(\boldsymbol{p})$ for $|\boldsymbol{p}| \to 0$, for instance if $E = A|\boldsymbol{p}|^{\alpha}$, where A and α are constants.

In particular, this should be true for the free non-relativistic particle where $E = (1/2m)\,p^2$. We shall show in the following sections that this is indeed the case. The non-relativistic free motion characterized by $\ddot{\boldsymbol{x}} = 0$ has the full 15-parameter Liouville group [isomorphic to the group O(2,4)] as an exact symmetry group! This can immediately be seen from the correspondence $\boldsymbol{x} \to \boldsymbol{x}$, $x^0 = ct \to y^0 = vt$, $v = p/m$. In addition, it is invariant under the special Galilei group $G_3(\boldsymbol{b})$ and the time translations $T_1(\tau)$. However we have $\boldsymbol{g} = v^{-1}\boldsymbol{n}$ and $E = \frac{1}{2}vp^0$, where $\boldsymbol{g}$ is the conserved "Galilei"-momentum associated with Galilei invariance, $\boldsymbol{n}$ is the conserved "Lorentz" momentum associated with the special "Lorentz"-transformations contained in the group O(2,4), and p^0 is the conserved quantity associated with the "y^0"-translations $y^0 \to y^0 + \bar{\tau}$.

Suppose now it is possible to give a quantum mechanical description of a free non-relativistical particle by means of a certain faithful unitary representation of the group O(2,4). Looking at the classical situation one is inclined to expect that the generators G^k, $k = 1, 2, 3$, and H of the special Galilei group and the time translations are contained in the enveloping algebra of the Lie-algebra of the O(2,4) according to the relations $G^k = mP^{-1}M^{0k}$ and $H = (2m)^{-1}PP^0$. This is indeed the case for particles with vanishing spin, but only for the physical projective representations of the Galilei group, not for the unphysical faithful ones! Thus, starting with the gauge properties of a free non-relativistic *classical* particle, we arrive at those "unusual" but physical representations of the Galilei group in a straightforward way. The case of non-vanishing spin is more complicated, however.

The paper is organized as follows: In sect. 2 we give the infinitesimal coordinate transformations induced by the orthochronous proper 15-parameter Liouville group. In sect. 3 the constants of motion associated with these transformations are listed. Sect. 4 contains the finite continuous and sect. 5 the discrete transformations. The stationary case $v = 0$ is discussed in sect. 6. Sect. 7 contains some remarks on the relation between unitary representations of the Liouville group and the projective unitary representations of the Galilei group. The problem of the invariance or non-invariance of interactions under dilatations is treated in sect. 8.

2. INFINITESIMAL TRANSFORMATIONS OF THE COORDINATES

We consider a free non-relativistic particle with the kinetic energy $E = (1/2m)p^2$. Its velocity is $\boldsymbol{v} = \partial E/\partial \boldsymbol{p} = \boldsymbol{p}/m = \mathrm{d}\boldsymbol{x}/\mathrm{d}t$. We define $y^0 = vt$ and call the set of points $y = (y^0, \boldsymbol{x})$ with the metric $(y, y) = (y^0)^2 - \boldsymbol{x}^2$ the "Galilei space". In the following we consider only infinitesimal transformations and assume $v \neq 0$. The case $v = 0$ will be discussed in a later section.

First we rewrite the transformations (1a)-(1d) for the coordinates y. We have

$$T_3(\boldsymbol{a}): \quad y^0 \to y^0, \qquad v \to v, \tag{3a}$$

$$T_1(\tau): \quad y^0 \to y^0 + v\tau, \qquad v \to v, \tag{3b}$$

$$R_3(\omega): \quad y^0 \to y^0, \qquad v \to v, \tag{3c}$$

$$G_3(\boldsymbol{b}): \quad \boldsymbol{x} \to \boldsymbol{x} + v^{-1}\boldsymbol{b}y^0, \quad y^0 \to y^0 + v^{-2}\,\boldsymbol{v}\cdot\boldsymbol{b}y^0,$$
$$v \to v + v^{-1}\,\boldsymbol{v}\cdot\boldsymbol{b}. \tag{3d}$$

In addition to the time translations $T_1(\tau)$ we define the y^0-translations

$$T_1(\bar{\tau}): \quad x \to x, \quad y^0 \to y^0 + \bar{\tau}, \quad t \to t + v^{-1}\bar{\tau}, \quad v \to v. \tag{3e}$$

The "special Lorentz"-transformations of the Galilei space are given by

$$\begin{aligned} N_3(\boldsymbol{u}): \ & \boldsymbol{x} \to \boldsymbol{x} - \boldsymbol{u} y^0, \qquad y^0 \to y^0 - \boldsymbol{u}\cdot\boldsymbol{x}, \\ & t \to t - v^{-1}\boldsymbol{u}\cdot\boldsymbol{x}, \quad v \to v. \end{aligned} \tag{3f}$$

We now come to those transformations of the Galilei space which correspond to the groups (2a) and (2b). We get:

$$\begin{aligned} D_1(\alpha): \quad & \boldsymbol{x} \to \boldsymbol{x} + \alpha \boldsymbol{x}, \quad y^0 \to y^0 + \alpha y^0, \\ & v \to v - \alpha v, \qquad t \to t + 2\alpha t. \end{aligned} \tag{3g}$$

$$\begin{aligned} SC_4(c): \quad & \boldsymbol{x} \to \boldsymbol{x} + (\boldsymbol{x}^2 - (y^0)^2)\boldsymbol{c} - 2(\boldsymbol{c}\cdot\boldsymbol{x})\boldsymbol{x}, \\ & y^0 \to y^0 - 2(\boldsymbol{c}\cdot\boldsymbol{x})y^0, \\ & t \to t - 4(\boldsymbol{c}\cdot\boldsymbol{x})t + 2(\boldsymbol{c}\cdot\boldsymbol{v})t^2, \\ & v \to v + 2(\boldsymbol{c}\cdot\boldsymbol{x})v - 2(\boldsymbol{c}\cdot\boldsymbol{v})t^2 v. \end{aligned} \tag{3h}$$

$$\begin{aligned} & \boldsymbol{x} \to \boldsymbol{x} + 2c^0 v t \boldsymbol{x}, \\ & y^0 \to y^0 + y^0 + 2c^0(y^0)^2 - c^0((y^0)^2 - \boldsymbol{x}^2), \\ & t \to t + c^0 v t^2 + c^0 v^{-1}\boldsymbol{x}^2, \\ & v \to v. \end{aligned} \tag{3i}$$

The transformations (3a)-(3i), except for (3b) and (3d), are the infinitesimal transformations of the 15-parameter orthochronous proper Liouville group of the Galilei space. The group itself is isomorphic to the group [4] SO(2,4).

3. THE CONSERVED QUANTITIES

The transformations (3a)-(3i) leave the action integral,

$$\int \mathrm{d}t \, \frac{1}{2m} \, (\mathrm{d}\boldsymbol{x}/\mathrm{d}t)^2 \, ,$$

of a free particle invariant (in some cases there remains an additional integral of a total differential, which does not change the conclusions). According to the well-known theorems [13] of Noether this implies the following conserved quantities:

$$\mathrm{T}_3(\boldsymbol{a}): \text{ Momentum } \boldsymbol{p}, \tag{4a}$$

$$\mathrm{T}_1(\tau): \text{ Energy } E = \frac{1}{2m} p^2, \tag{4b}$$

$$\mathrm{R}_3(\omega): \text{ Angular momentum } \boldsymbol{m} = \boldsymbol{x} \times \boldsymbol{p}, \tag{4c}$$

$$\mathrm{G}_3(\boldsymbol{b}): \text{ Galilei momentum } \boldsymbol{g} = m\boldsymbol{x} - \boldsymbol{p}t, \tag{4d}$$

$$\mathrm{T}_1(\bar{\tau}): \text{ Modulus of momentum } p^0 = +(\boldsymbol{p}^2)^{\frac{1}{2}}, \tag{4e}$$

$$\mathrm{N}_3(\boldsymbol{u}): \text{ Lorentz momentum } \boldsymbol{n} = \boldsymbol{x}p^0 - vt\boldsymbol{p}, \tag{4f}$$

$$\mathrm{D}_1(\alpha): \text{ Dilatation momentum } s = 2Et - \boldsymbol{x}\cdot\boldsymbol{p}, \tag{4g}$$

$$\mathrm{SC}_4(c): \begin{array}{l}\text{Bessel-Hagen momentum } \boldsymbol{h} = 2\boldsymbol{x}s - (v^2t^2 - \boldsymbol{x}^2)\boldsymbol{p}, \\ \text{modulus of the Bessel-Hagen momentum} \\ h^0 = 2vts - (v^2t^2 - \boldsymbol{x}^2)p^0 = +(\boldsymbol{h}^2)^{\frac{1}{2}}. \end{array} \tag{4h}$$

We notice [4] the following relations:

$$\boldsymbol{h}\times\boldsymbol{p} = 2s\boldsymbol{m}, \quad p^0\boldsymbol{h} - h^0\boldsymbol{p} = 2s\boldsymbol{n}, \quad h^0p^0 - \boldsymbol{h}\cdot\boldsymbol{p} = 2s^2. \tag{5}$$

The relations (5) determine the quantities $\boldsymbol{m}$, $\boldsymbol{n}$ and s in terms of the two vectors $\boldsymbol{p}$ and $\boldsymbol{h}$, except for a relative sign.

In addition we have

$$\boldsymbol{g} = v^{-1}\boldsymbol{n}, \quad v = p^0/m, \tag{6}$$

$$E = \tfrac{1}{2}vp^0. \tag{7}$$

Thus, all the known conserved quantities of a free non-relativistic particle can be generated by $\boldsymbol{p}$ and $\boldsymbol{h}$!

It should be mentioned that the mass m is an invariant under the full non-relativistic Liouville group. This is not so in the relativistic case [4].

4. DISCRETE TRANSFORMATIONS

Under space reflections P and time reversal T the quantities s, $\boldsymbol{h}$ and h^0 transform as follows:

$$\mathrm{P}: \quad s \to s, \quad \boldsymbol{h} \to -\boldsymbol{h}, \quad h^0 \to h^0,$$

$$\mathrm{T}: \quad s \to -s, \quad \boldsymbol{h} \to -\boldsymbol{h}, \quad h^0 \to h^0.$$

The property that $\boldsymbol{p}$ and $\boldsymbol{h}$ are invariant under the product PT, but $\boldsymbol{m}$, $\boldsymbol{n}$ and s change sign, is, of course, closely related to the fact that $\boldsymbol{m}$, $\boldsymbol{n}$ and s are determined by $\boldsymbol{h}$ and $\boldsymbol{p}$ only up to a sign.

In addition to P and T we have the discrete "length inversion" [4, 9] R with the properties:

$$\mathrm{R}: \quad \boldsymbol{m} \to \boldsymbol{m}, \quad \boldsymbol{n} \to \boldsymbol{n}, \quad s \to -s, \quad \boldsymbol{p} \to \boldsymbol{h}, \quad p^0 \to h^0, \quad \boldsymbol{h} \to \boldsymbol{p}, \quad h^0 \to p^0. \tag{8}$$

The mass m is invariant under R.

In the Galilei space R has the form [14]

$$y^0 \to y^{0\prime} = v't' = vt/(\boldsymbol{x}^2 - v^2t^2)\,, \qquad v' = p'/m = h^0/m\,,$$

$$\boldsymbol{x} \to \boldsymbol{x}' = \boldsymbol{x}/(\boldsymbol{x}^2 - v^2t^2)\,. \tag{9}$$

The fact [4] that the translations $T_3(\boldsymbol{a})$ and $T_1(\bar{\tau})$, the time inversion T, and the length inversion R generate the full 15-parameter Liouville group underlines the importance of the discrete group R.

5. THE FINITE TRANSFORMATIONS

It was already stressed in the introduction - and discussed in detail in refs. [4, 9] - that the finite transformations $SC_4(c)$ and R cannot be defined satisfactorily in the Minkowski or Galilei space. One has to introduce homogeneous coordinates $y^0 = \eta^0/\kappa$, $x^i = \eta^i/\kappa$, $i = 1, 2, 3$, where κ is a Poincaré invariant unit of length and $\eta = (\eta^0, \boldsymbol{\eta})$ characterizes the position. If we define the spurious coordinate λ by $\kappa\lambda = (\eta^0)^2 - \boldsymbol{\eta}^2$ then R appears to be the discrete gauge transformation $\eta \to \eta$, $\kappa \to -\lambda$, $\lambda \to -\kappa$. Thus, the interpretation and the formalism is exactly the same as in refs. [4, 9], and we shall not repeat them here.

6. THE CASE $v = 0$

If $v = 0$ then $y^0 = 0$. In this case the proper 15-parameter Liouville group degenerates into the proper 10-parameter Liouville group of the three-dimensional Euklidean space with coordinates $\boldsymbol{x}$. It is isomorphic to the group SO(1,4). Thus the Euclidean group (R_3 plus T_3), combined with the gauge groups D_1 and SC_3 of the Euclidean space, yield the group SO(1,4). Its physical interpretation is the same as that of the group SO(2,4) in the Galilei or Minkowski space.

As the Galilei group changes the velocity v, it cannot be incorporated into the framework of the Liouville group of the Euclidean space.

In order to illustrate the physical significance of the dilatations and the special Liouville group $SC_3(c)$ in the case $y^0 = 0$, i.e. for either $v = 0$ or $t = 0$ (stationary systems!), we shall discuss some details now.

The 10-parameter proper Liouville group of the three-dimensional Euclidean space is given by

$$T_3(\boldsymbol{a}):\quad x^i \to x^i + a^i, \quad i = 1, 2, 3, \tag{10a}$$

$$R_3(r):\quad x^i \to r^{ik}x^k, \quad r^{ik}r^{lk} = \delta^{il}, \tag{10b}$$

$$D_1(\alpha):\quad x^i \to e^{\alpha}x^i, \tag{10c}$$

$$SC_3(c):\quad x^i \to RT_3(c)Rx^i = \frac{1}{\sigma(\boldsymbol{x})}(x^i + c^i\boldsymbol{x}^2)\,,$$

$$Rx^i = x^i/\boldsymbol{x}^2$$

$$\sigma(\boldsymbol{x}) = 1 + 2\boldsymbol{c}\cdot\boldsymbol{x} + \boldsymbol{c}^2\boldsymbol{x}^2. \tag{10d}$$

Eq. (10d) shows that the special Liouville group is isomorphic to the translations.

Since we are dealing with non-linear transformations, we have to define distances by the differential form $ds^2 = dx^i dx^i$. Because of $ds^2 \to (1/\sigma(\boldsymbol{x}))^2 ds^2$ in the case of $SC_3(\boldsymbol{c})$ we interpret the special Liouville group as a group which induces position dependent ("local") geometrical gauge transformations in the sense that a given length ds at a point $\boldsymbol{x}$ is mapped onto another one which differs from the first one by the position dependent factor $1/\sigma(\boldsymbol{x})$.

All this becomes more transparent if we introduce the unit of length which is being employed at each point explicitly. The introduction of these new coordinates is necessary anyhow, because one cannot have a one-to-one mapping in $\boldsymbol{x}$-space as far as the groups $SC_3(\boldsymbol{c})$ and R are concerned. For instance, the point $\boldsymbol{x} = 0$ has no image in $\boldsymbol{x}$-space with respect to the mapping R.

We define a Euclidean invariant unit of length κ by the equations

$$x^i = \eta^i/\kappa, \qquad i = 1, 2, 3. \tag{11}$$

The numbers η^i characterize the location in space and the quantity κ the unit of length employed at this location (note that (η^i, κ) and $\beta(\eta^i, \kappa)$, $\beta \neq 0$ and constant, are equivalent. They correspond to the same location.).

In addition we define a spurious coordinate λ by the equation

$$\lambda\kappa = \eta^i\eta^i. \tag{12}$$

The groups $T_3(\boldsymbol{a})$, $R_3(\boldsymbol{r})$, $D_1(\alpha)$ and R induce the following transformations in the space of these new coordinates:

$$\begin{aligned}
T_3(\boldsymbol{a}):\quad & \eta^i \to \eta^i + a^i\kappa, \quad i = 1, 2, 3, \\
& \kappa \to \kappa, \\
& \lambda \to \lambda + 2a^i\eta^i + \boldsymbol{a}^2\kappa,
\end{aligned} \tag{13a}$$

$$R_3(\boldsymbol{r}):\quad \eta^i \to r^{ik}\eta^k, \quad \kappa \to \kappa, \quad \lambda \to \lambda \tag{13b}$$

$$D_1(\alpha):\quad \eta^i \to \eta^i, \quad \kappa \to e^{-\alpha}\kappa, \quad \lambda \to e^{\alpha}\lambda, \tag{13c}$$

$$R\quad:\quad \eta^i \to \eta^i, \quad \kappa \to \lambda, \quad \lambda \to \kappa. \tag{13d}$$

The transformations induced by the group $SC_3(\boldsymbol{c})$ can be constructed from (13a) and (13d), because its elements are given by $RT_3(\boldsymbol{c})R$.

The gauge character of R is evident from eq. (13d): The position, characterized by η^i, stays the same but the unit of length is changed.

The above transformations leave the form $\lambda\kappa - \eta^i\eta^i$ unchanged. This means that the 10-parameter proper Liouville group of the Euclidean space is isomorphic to the "orthochronous proper Lorentz" group SO(1,4) in five dimensions (put $\kappa = \eta^5 - \eta^4$, $\lambda = \eta^5 + \eta^4$).

The length element ds in terms of the new coordinates is given by

$$ds^2 = \frac{1}{\kappa^2}(d\eta^i\, d\eta^i - d\kappa\, d\lambda) \geqslant 0.$$

This form is positive definite because of the subsidiary condition $2\eta^i d\eta^i - \lambda\, d\kappa - \kappa\, d\lambda = 0$ which follows from eq. (12).

The full Liouville group of the three-dimensional Euclidean space consists of two pieces, which are mapped onto each other either by the space reflection P or by the length inversion R, but the full group O(1,4) consists of four pieces, characterized by the sign of the determinant of the transformation matrix (the sign is -1 for P and R) and the sign of the coefficient $b^{5}_{\cdot 5}$ of η^5 (+1 for P and R). However, because the relation (11) is not changed if we reverse the sign of η^i, η^4 and η^5, the piece (det = -1, sign $b^{5}_{\cdot 5}$ = -1) of the group O(1,4) is mapped onto the same piece of the full Liouville group as the piece (det = +1, sign $b^{5}_{\cdot 5}$ = +1). In the same manner the pieces (det = -1, sign $b^{5}_{\cdot 5}$ = +1) and (det = +1, sign $b^{5}_{\cdot 5}$ = -1) are mapped onto the second piece of the full Liouville group.

The description of physical quantities in terms of tensors and spinors of the group SO(1,4) is analogous to the relativistic case [4] and will not be treated here.

7. UNITARY REPRESENTATIONS IN THE CASE OF VANISHING SPIN

We have seen in sects. 2 and 3 that the motion of a free non-relativistic particle is invariant under the 15-parameter Liouville group. In addition we have invariance under time translations and Galilei transformations. The conserved quantities E and $\boldsymbol{g}$ associated with these latter groups can be expressed in terms of the conserved quantities associated with the Liouville group.

We now turn - very tentatively - to a few problems associated with unitary representations of the Liouville group and ask for their relations to the unitary representations of the 10-parameter Galilei group mentioned in the introduction. For simplicity we consider only the case with vanishing spin. The case of non-vanishing spin is definitely more complicated and not a trivial generalization of the one to be discussed here.

We start with a unitary representation of the proper 15-parameter Liouville group in the Hilbert space of functions $\varphi(\boldsymbol{p})$ of the momenta $\boldsymbol{p}$, $p^0 = +(\boldsymbol{p}^2)^{\frac{1}{2}}$, with the scalar product

$$(\varphi_1, \varphi_2) = \int \frac{d^3\boldsymbol{p}}{2p^0} \varphi_1^*(\boldsymbol{p})\, \varphi_2(\boldsymbol{p}). \tag{14}$$

The representation we are going to consider contains a representation of the "Poincaré" group with spin zero and vanishing "rest mass", i.e. $(p^0)^2 - p^2 = 0$. The Hermitian generators of the infinitesimal transformations of the 15-parameter Liouville group are denoted as follows: $T_3(\boldsymbol{a}) : P^i = -P_i$, $T_1(\bar{\tau}) : P^0 = P_0$, $R_3(\omega) : M^{ik}$, $i, k = 1, 2, 3$, $N_3(\boldsymbol{u}) : M^{0i}$, $i = 1, 2, 3$, $D_1(\alpha) : D$, $SC_4(c) : K^i$, K^0. In the space of functions we are considering, these generators have the form [10]:

$$P^0 = p^0 , \quad P^i = p^i , \tag{15a}$$

$$M^{\mu\nu} = i\left(p^\mu \frac{\partial}{\partial p_\nu} - p^\nu \frac{\partial}{\partial p_\mu}\right), \qquad \mu, \nu = 0, 1, 2, 3, \tag{15b}$$

$$D = i(p^{\nu} \frac{\partial}{\partial p^{\nu}} + 1), \tag{15c}$$

$$K^{\mu} = -2 \frac{\partial}{\partial p_{\mu}} - 2p^{\nu} \frac{\partial}{\partial p^{\nu}} \frac{\partial}{\partial p_{\mu}} + p \quad \frac{\partial}{\partial p^{\nu}} \frac{\partial}{\partial p_{\nu}}, \quad \mu = 0,1,2,3, \tag{15d}$$

In the following we shall not need the explicit expressions for D and K^{μ}.

In sect. 3 we had the relations $E = (p/2m)p^{0}$ and $\boldsymbol{g} = (m/p)\boldsymbol{n}$. In a representation these classical conserved quantities correspond to the generators of the infinitesimal transformations. If H and G^{j}, $j = 1, 2, 3$, are the generators of the time translations and the Galilei transformations, we therefore try

$$H = (p/2m)\,p^{0} = \frac{p^2}{2m} = E\,, \tag{16}$$

$$\begin{aligned} G^{j} &= (m/p)M^{0j} \\ &= i^{-1}\,(m \frac{\partial}{\partial p_j} + \frac{m}{p}\,p^{j} \frac{\partial}{\partial p^{0}})\,. \end{aligned} \tag{17}$$

If we put $\tilde{\varphi}(E,\boldsymbol{p}) \equiv \varphi(p^{0},\boldsymbol{p})$, we have $\partial\varphi/\partial p^{0} = (p^{0}/m)\,\partial\tilde{\varphi}/\partial E$, i.e. with respect to the functions $\tilde{\varphi}$ the generator G^{j} has the form

$$G^{j} = i^{-1}\,(m \frac{\partial}{\partial p_j} + p^{j} \frac{\partial}{\partial E})\,. \tag{18}$$

This is exactly the form the generators of the Galilei transformations have in the "physical" representation [3] for spin zero and mass m. The "Lorentz" invariant measure $d\Omega_0 = d^3\boldsymbol{p}\; dp^0\,\theta(p^0)\,\delta(p_0^2 - \boldsymbol{p}^2)$ has to be replaced by $(2p/m)\,d\Omega_0 = d^3\boldsymbol{p}\,dE\,\delta\{E - (p^2/2m)\}$ and we have

$$(\tilde{\varphi}_1, \tilde{\varphi}_2) = \int d^3p\,dE\,\delta(E - \frac{p^2}{2m})\,\tilde{\varphi}_1^{*}(p)\,\tilde{\varphi}(p_2)\,.$$

It is very interesting that by starting from the classical conserved quantities of the free non-relativistic particle, we arrive immediately at a "physical" representation of the Galilei group, not at an "unphysical" faithful one! This is a neat future which may help to explain the "strange" situation as far as the physical significance of the different types of unitary representations of the Galilei group are concerned.

8. INTERACTIONS

The symmetry group O(2,4) of the free non-relativistic particle is generally broken if interactions are taken into account. We shall discuss this symmetry breaking for unitary representations systematically in a second paper [15]. In this section we shall deal with some features pertaining to the classical case:

(i) Consider the Hamilton function $H = \frac{1}{2}\boldsymbol{p}^2 + V(\boldsymbol{x})$ of a particle in a potential $V(\boldsymbol{x})$. The time derivative $\mathrm{d}s/\mathrm{d}t$ of the dilatation momentum $s = 2Et - \boldsymbol{r}\cdot\boldsymbol{p}$ is given by

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 2V(\boldsymbol{x}) + \boldsymbol{x}\cdot \mathrm{grad}\ V(\boldsymbol{x}).$$

This means that the quantity s is a constant of motion only if the potential $V(\boldsymbol{x})$ either vanishes identically or is homogeneous of degree -2. A rotationally invariant potential of this type is $V = -A/\boldsymbol{x}^2$. This potential modifies the centrifugal potential. By using explicit expressions [16] for $\boldsymbol{x} = \boldsymbol{x}(t)$ and $\boldsymbol{p}(t)$ one can verify that the dilatation momentum is indeed a constant of motion for this potential.

(ii) In order to illustrate the reason why most of the non-relativistic systems with interactions are not invariant even under dilatations, we consider a particle with mass m in the gravitational field of a particle with mass M. The Hamilton function is $H = (1/2m)p^2 - G(mM/r) = E$, $r = |\boldsymbol{x}|$. Since the fixed masses m and M are considered to be invariant under the dilatations $x \to \mathrm{e}^{\alpha} x$, the time t has to transform as $t \to \mathrm{e}^{2\alpha} t$ in order to make the kinetic term of the action integral invariant. But then the interaction term of the action integral is not invariant under dilatations for the following reasons: It is natural to use the units length, mass and action in non-relativistic systems. In the framework of these units the gravitational constant G has the dimension $(\mathrm{length})^{-1}(\mathrm{mass})^{-3}(\mathrm{action})^2$. Since G is a fixed constant for the system under consideration, it does not change under dilatations!

The deeper reason for the non-invariance under dilatations is, therefore, that the coupling constant G contains a fixed length with respect to the motion of the two particles of the system. A theory about the physical origin of such a fixed length would probably shed new light on the breaking of dilatation invariance (whether it is of cosmic or atomic origin, for instance!).

On the other hand it is of great importance that - contrary to the non-relativistic situation - many important relativistic systems like quantum electrodynamics, etc., have Liouville invariant interaction Lagrangeans [4, 17]. For these systems the symmetry is broken by the kinetic mass terms.

(iii) The non-invariance of many non-relativistic action integrals under dilatations does not mean that the corresponding equations of motion are not invariant. If we have

$$m\,\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2} = -\mathrm{grad}\ V(\boldsymbol{x}), \tag{19}$$

and if $V(\boldsymbol{x})$ is homogeneous of degree β, then the equation of motion (19) is invariant under dilatations $\boldsymbol{x} \to \mathrm{e}^{\alpha}\boldsymbol{x}$, if the time t transforms as $t \to \mathrm{e}^{\alpha(1-\frac{1}{2}\beta)}\,t$. This invariance may be quite useful [16], but it does not lead to new conservation laws.

(iv) The dilatation momentum $s = 2Et - \boldsymbol{r}\cdot\boldsymbol{p}$ is of interest even if it is not conserved: If the potential V is homogeneous of degree β, then the time derivative of s is

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 2E - \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{x}\cdot\boldsymbol{p}) = (2+\beta)\,V(\boldsymbol{x}). \tag{20}$$

If the system is such that the quantity $|\boldsymbol{x}\cdot\boldsymbol{p}|$ is bounded from above, we can calculate the time average of eq. (20). With the definition

$$\bar{F} = \lim_{T\to\infty} \frac{1}{T} \int_0^T F(t)\,\mathrm{d}t,$$

we get $\overline{\mathrm{d}s/\mathrm{d}t} = 2E = (2+\beta)\,\overline{V(\boldsymbol{x})}$, or

$$\overline{V(\boldsymbol{x})} = \frac{2E}{2+\beta}. \tag{21}$$

Eq. (21) is the well-known virial theorem [18].

(v) As the dilatation momentum s and the Bessel-Hagen momenta $\boldsymbol{h}$ and h^0 are not familiar to many physicists, it seems to be worthwhile to illustrate their intuitive meaning. In order to do this, we consider the non-relativistic elastic scattering of two particles of the same mass m, assuming merely the conservation laws associated with the 10-parameter Galilei group, the dilatations, and the special Liouville group. The existence of a Hamilton function is not required. The following discussion is analogous to that of the extreme relativistic case of ref. [4].

We assume the asymptotic motions of the two particles at $t\to-\infty$ and $t\to+\infty$ can be characterized by the free motions

$$\boldsymbol{x}_i = (\boldsymbol{p}_i/m)t + \boldsymbol{a}_i, \quad i = 1,2, \quad t\to-\infty, \tag{22a}$$

$$\boldsymbol{x}_i' = (\boldsymbol{p}_i'/m)t + \boldsymbol{a}_i', \quad i = 1,2, \quad t\to+\infty. \tag{22b}$$

Before the scattering we have in the c.m. system $\boldsymbol{p}_1 = \boldsymbol{p} = -\boldsymbol{p}_2$, $\boldsymbol{x}_1 + \boldsymbol{x}_2 = \boldsymbol{X} = \text{const}$. We choose $\boldsymbol{X} = 0$ and have $\boldsymbol{a}_1 = \boldsymbol{a} = -\boldsymbol{a}_2$. Because of the interaction the quantities $\boldsymbol{p}_i'$, $i = 1,2$, and $\boldsymbol{a}_i'$, $i = 1,2$, are in general different from $\boldsymbol{p}_i$ and $\boldsymbol{a}_i$. We consider now the constraints imposed by the conservation laws mentioned above:

Momentum conservation gives $\boldsymbol{p}_1' = \boldsymbol{p}' = -\boldsymbol{p}_2'$. From energy conservation we have $E_1 + E_2 = 2E = E_1' + E_2'$. Conservation of the total Galilei momentum $\boldsymbol{g}_1 + \boldsymbol{g}_2$ yields $\boldsymbol{a}_1' = \boldsymbol{a}' = -\boldsymbol{a}_2'$. Angular momentum conservation gives

$$\boldsymbol{m}_1 + \boldsymbol{m}_2 = 2\boldsymbol{a}\times\boldsymbol{p} = \boldsymbol{m}_1' + \boldsymbol{m}_2' = 2\boldsymbol{a}'\times\boldsymbol{p}'. \tag{23}$$

From this it follows that $a\sin\gamma = a'\sin\gamma'$, where γ and γ' are the angles between $\boldsymbol{a}$ and $\boldsymbol{p}$ and $\boldsymbol{a}'$ and $\boldsymbol{p}'$ respectively.

Angular momentum conservation means that the "vertical distance" of the asymptotic straight lines of motion from the origin - the impact parameter - is the same before and after the scattering. It does not say anything about the "parallel" distance $a\cos\gamma$. In general we shall have $a\cos\gamma \neq$ $\neq a'\cos\gamma'$, because the interaction slows down or accelerates the particles in the interaction region (positive or negative time delay).

The new feature now is that the conservation of the total dilatation momentum $s_1 + s_2$ forbids such a time delay! For we have

$$s_1 + s_2 = -2\boldsymbol{a}\cdot\boldsymbol{p} = s_1' + s_2' = -2\boldsymbol{a}'\cdot\boldsymbol{p}', \tag{24}$$

or $a \cos\gamma = a' \cos\gamma'$. Combined with eq. (23) this implies $a' = a$ and $\gamma' = \gamma$.

The above result means that in an elastic scattering which conserves not only the usual ten quantities but also the total dilatation momentum, the vector $\boldsymbol{a}$ can only be rotated, the angle of rotation being the scattering angle. Therefore, the cross section can depend only on the scattering angle in a non-trivial way, its energy dependence is determined because $\boldsymbol{a}'$ depends only on the scattering angle. This is, of course, a severe restriction, and we expect the conservation of the total dilatation momentum only in limiting or approximate cases.

Finally, it turns out that the conservation of the total Bessel-Hagen momenta $\boldsymbol{h}_1 + \boldsymbol{h}_2$ and $h_1^0 + h_2^0$ is fulfilled automatically in elastic scattering, if the sum of the dilatation momenta is conserved.

(vi) It has already been mentioned [14] that the quantities discussed in sect. 3 are conserved for any elementary excitation with a dispersion law $E = Ap^{\alpha}$. Such excitations play an important role in low-temperature solid-state physics (phonons, magnons etc.). If the interaction between two such elementary excitations is approximately dilatation invariant, the cross section for an elastic scattering of such excitations off each other will have the approximate form [4] $\mathrm{d}\sigma/\mathrm{d}\Omega = E^{-2} A(\theta)$, where E is the c.m. energy and A depends only on the scattering angle θ. If $E \to 0$ for T(temperature) $\to 0$, then the cross section diverges. In a very intuitive sense this means that long-range correlations become important. Something like this seems to happen for many-particle systems at vanishing absolute temperatures.

However, we want to emphasize that these remarks are mere speculations and that a more detailed analysis is certainly necessary before one can say more about the importance of dilatations and the special Liouville group for low-temperature physics.

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