Symmetries in Physics (1600-1980)

Proceedings of the
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PREFACE

These are the Proceedings of an international meeting on the History of Symmetries in Physics, which we are pleased to present as members of the Organizing Committee.

The suggestion of such a meeting was issued by the Rectorate of the Universitat Autònoma de Barcelona, in view of its interest to promete within the University the work on History of Science, and of its recent convention with Sant Feliu de Guíxols' Townhall for cultural collaboration. This suggestion took a concrete form in successive binate conversations between us (M.G.D.-L.M. on December 1982, and M.G.D.-A.H. on February 1983, both at Universitat Autònoma de Barcelona; A.P.-L.M. on April 1983 at Rockefeller University). The main point was to choose a physics topic of established interest for today's physicists, whose evolution would be throughly discussed since the very beginning of modern physics. This discussion should reach an interdisciplinary level in which the points of view of both, physicists and historians of physics would be represented and respected in a harmonious equilibrium. Distinguished physicists involved in the introduction of different kinds of symmetries and/or reputed scholars working in the history of modern physics were invited. In fact a true atmosphere of mutual comprehension and agreable collaboration was felt in Sant Feliu, thus overcoming the language barriers of these two cultures. If any difference in opinion arose everybody tried to do his best to discuss it in depth.

The different aspects of physical symmetries were developed in 22 lectures with short discussion and 4 round tables (around 30 videotape hours). A session of seminars or short expositions of communications presented by the participants was also organized. This material is now edited in these Proceedings in the same order it was presented, during the six working days of the meeting (the only exception is the round table in section 27 which will be explained further on). This order should reflect in some way the chronological thematic order in which these symmetries were introduced in the history of physics, the six lectures and the short seminar session of the first two days were namely devoted to prequantic symmetry, from the physics of Galileo and Newton, through Euler, the dynamists and the Göttingen School, to the introduction of relativity and the first ideas on symmetry breaking. The second day concluded with a round table directed by one of us (A.H.) in which different aspects of these classical symmetries were spontaneously related to the most modern ones. The third day was dedicated to quantum symmetries: Bose and Fermi statistics, symmetries of matter and light, symmetry ideas of Bohr, Einstein, Heisenberg and Pauli. On the fourth day we dealt with energy conservation, cosmological symmetries and the discrete symmetries: C and T introduction, P and CP violation. The scheduled round tables on the topics of these two days had to be cancelled due to lack of time, nevertheless some lectures, for instance those of Edoardo Amaldi on Fermi statistics, Valentine Telegdi on parity violation and Val Fitch on CP violation

References

BERNARDINI, PAOLO and ELISABETTA DONINI
1980 «Perchè fu Einstein il tramite tra de Broglie e Schrödinger»; *Proceedings* of the International Conference *The Recasting of Science between the Two World Wars* (Firenze-Roma 1980) in press.

DE BROGLIE, LOUIS
1973 «The Beginning of Wave Mechanics» in W.C. Price, S.S. Chissick, T. Ravensdale, eds., *Wave Mechanics The First Fifty Years* (Butterworths, London 1973) 12-18.

DIRAC, PAUL A.M.
1926 «On the Theory of Quantum Mechanics»; *Proceedings of the Royal Society of London A112*, 661-677.

DONINI, ELISABETTA
1982 *Il caso dei quanti - Dibattito in fisica e ambiente storico, 1900-1927* (Clup-Clued, Milano 1982).

EINSTEIN, ALBERT
1905a «Über einen die Erzeugung und Verwandlung des Lichtes betreffenden heuristischen Gesichtspunkt»; *Annalen der Physik 17*, 132-148.
1905b «Zur Elektrodynamik bewegter Körper»; *Annalen der Physik 17*, 891-921.
1909 «Zum gegenwärtingen Stand des Strahlungsproblems»; *Physikalische Zeitschrift 10*, 185-193.
1919 «My Theory»; *The London Times* (Nov. 28, 1919) 13.
1924 «Quantentheorie des einatomigen idealen Gases»; *Preussische Akademie der Wissenschaften, Sitzungsberichte* (1924), 261-267.
1925 «Quantentheorie des einatomigen idealen Gases. Zweite Abhandlung», *Preussische Akademie der Wissenschaften, Sitzungsberichte* (1925), 3-14.

HEISENBERG, WERNER and WOLFGANG PAULI
1929 «Zur Quantendynamik der Wellenfelder»; *Zeitschrift für Physik 56*, 1-61.
1930 «Zur Quantentheorie der Wellenfelder. II»; *Zeitschrift für Physik 59*, 168-190.

JORDAN, PASCUAL
1927 «Zur Quantenmechanik der Gasentartung»; *Zeitschrift für Physik 44*, 473-480.

PAIS, ABRAHAM
1982 *«Subtle is the Lord» The Science and the Life of Albert Einstein* (Clarendon Press, Oxford 1982).

PAULI, WOLFGANG
1921 «Relativitätstheorie»; *Encyclopädie der Mathematischen Wissenschaften 5*, 539-775; Italian translation: *Teoria della relatività* (Boringhieri, Torino 1958).
1927 «Zur Quantenmechanik des magnetischen Elektrons»; *Zeitschrift für Physik 43*, 601-623.
1946 «Exclusion Principle and Quantum Mechanics», Nobel lecture delivered Dec. 13, 1946; reprinted in W. Pauli, *Collected Scientific Works* (edited by R. Kronig and V.F. Weisskopf), (Interscience Pub., New York 1964) vol. 2, 1080-1096.

SOMMERFELD, ARNOLD
1919 *Atombau und Spektrallinien* (Vieweg, Braunschweig 1919).

TONIETTI, TITO
1980 «La meccanica quantistica nel contesto matematico: cambiamenti di punti di vista e qualche aspetto istituzionale»; in M. de Maria *et al.*, *Fisica & Società negli anni '20* (Clup-Clued, Milano 1980) 35-91.
1982 «A research proposal to study the formalist and intuitionist mathematicians of the Weimar Republic»; *Historia Mathematica 9*, 61-64.

WEYL, HERMANN
1918 *Raum-Zeit-Materie* (Berlin 1918); English translation of the fourth edition (1921): *Space-Time-Matter* (Dover, New York 1952).
1928 *Gruppentheorie und Quantenmechanik* (Teubner, Leipzig 1928); English translation of the second edition (1931): *The theory of Groups and Quantum Mechanics* (Dover, New York 1931).
1939 *The Classical Groups. Their Invariants and Representations* (Princeton U.P., Princeton 1939).
1952 *Symmetry* (Princeton U.P., Princeton 1952).

WIGNER, EUGENE P.
1931 *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren* (Vieweg & Sohn, Braunschweig 1931; English translation: *Group Theory and its application to the quantum mechanics of atomic spectra* (Academic Press, New York and London, 1959).

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THE CONTRIBUTIONS OF EMMY NOETHER, FELIX KLEIN AND SOPHUS LIE TO THE MODERN CONCEPT OF SYMMETRIES IN PHYSICAL SYSTEMS

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1. Introduction

> Warum aber überhaupt eine historische Betrachtungsweise von Dingen der Mathematik oder der Naturwissenschaften?... Ich glaube, daß wir heute mehr denn je eine solche historische Einstellung brauchen. Außerordentlich viel hängt für unsere Wissenschaften davon ab, ob und wie ihre Vertreter es verstehen, sich selbst und den Kreis ihrer Wirksamkeit als Glieder einer großen Entwicklungsreihe zu betrachten, und in welchem Maße sie imstande sind, aus dem Bewußtsein dieser Zusammenhänge für Gegenwart und Zukunft zu lernen.
> Richard Courant[1] (1888-1972)

On July 7 of the year 1918 Felix Klein presented the paper[2] «Invariante Variationsprobleme» by Emmy Noether at a session of the *Königliche Gesellschaft der Wissenschaften* (Royal Society of the Sciences) in Göttingen. Although this paper is a milestone in the history of the relation between symmetries and conservation laws in physics, it took more than 3 decades till its importance was —slowly— recognized. I suspect —perhaps unfairly so— that even in recent years only a few of those authors who quote Noether's work or refer to her «theorem» had a chance to see or study the original publication. This is mainly due to the fact that not many libraries posess(ed) the *Nachrichten von der Königlichen Gessellschaft der Wissenschaften zu Göttingen*, where the paper was published. As this year the collected papers of E. Noether appeared,[3] it is no longer necessary to get hold of a copy of the *Göttinger Nachrichten* from 1918! Noether's paper incorporates in a unique way different branches of mathematics and mathematical physics, namely:

i. Algebraic and differential invariant theory,
ii. Riemannian geometry and the calculus of variations in the context of general relativity, mechanics and field theory.
iii. Group theory, especially Lie's theory for solving or reducing differential equations by means of their invariance groups.

It is the aim of the following discussion to describe – briefly and incompletely – the historical background for these ingredients of Noether's work and to sketch the roles which Felix Klein, Sophus Lie and a few others played in the developments which let to our insights into the relations between the symmetry properties of a physical system and its conservation laws!

2. Emmy Noether's two Theorems

> The nature of the connection between symmetries and the existence of conserved quantities is an intriguing physical problem. The theory of this connections, as it appears in classical

1 COURANT 1926. Here is an attempt to translate: «But why indeed should we consider mathematical or scientific things in a historical way?... I believe that today more than ever we need such a historical viewpoint. For our sciences a great deal depends on whether their representatives are able to see themselves and the sphere of their activity as elements in a long series of development and to what extent they are able to learn for the present and the future from the awareness of these interrelationships.» As to Courant see: REID 1976.
2 NOETHER 1918b.
3 See NOETHER 1918b.

physics, constitutes one of the most beautiful chapters of mathematical physics. The fundamental work on this problem was done by Emmy Noether in 1918.
Andrzej Trautman[4]

Noether's work is of paramount importance to physics and the interpretation of fundamental laws in terms of group theory.
Feza Gürsey[5]

Before going into the historial background of Noether's paper let me first state without proof the *two* theorems which made her so famous among the physicists:

Suppose we have n fields $\varphi^i(x)$, $i=1,\dots,n$, depending on m variables $x=(x^1,\dots,x^m)$. The field equations are the Euler-Lagrange equations*

$$E_i(\varphi):=\frac{\partial L}{\partial\varphi^i}-\partial_\mu\frac{\partial L}{\partial(\partial_\mu\varphi^i)}=0$$

of the action integral

$$A=\int_G dx^1\dots dx^m L\,(x,\varphi^1,\dots,\varphi^n,\partial_\mu\varphi^1,\dots,\partial_\mu\varphi^n).$$

If

$$x^\mu\to\hat{x}^\mu=x^\mu+\delta x^\mu,$$

$$\varphi^i(x)\to\hat{\varphi}^i(\hat{x})=\varphi^i(x)+\delta\varphi^i=\varphi^i(x)+\bar{\delta}\varphi^i+\partial_\mu\varphi^i\delta x^\mu$$

are infinitesimal «variations» of the quantitites x^μ and φ^i, then one obtains for δA in lowest order of δx^μ and $\delta\varphi^i$:

$$\begin{aligned}\delta A&=\int_{\hat{G}} d\hat{x}^1\dots d\hat{x}^m\, L[\hat{x},\hat{\varphi}(\hat{x}),\partial\hat{\varphi}(\hat{x})]\\&\quad-\int_G dx^1\dots dx^m\, L[x,\varphi(x),\partial\varphi(x)]\\&=\int_G dx^1\dots dx^m\,[E_i(\varphi)\,\bar{\delta}\varphi^i+\partial_\mu B^\mu(x,\varphi,\partial\varphi,\delta x,\delta\varphi)],\end{aligned}\tag{1}$$

where the quantitites B^μ, $\mu=1,\dots,m$, are linear in δx^μ and $\delta\varphi^i$. From this expression for δA, E. Noether derived the following two theorems:

I. If the action integral A is invariant under an r-parameter Lie transformation group

$$x^\mu\to\hat{x}^\mu=f^\mu(x,\varphi;a^1,\dots,a^r),$$

$$\varphi^i(x)\to\hat{\varphi}^i(\hat{x})=F^i(x,\varphi;a^1,\dots,a^r),$$

4 TRAUTMAN 1967.
5 Feza Gürsey, quoted by N. Jacobson in his introduction in NOETHER *Papers* 23-25.

* In the following the Einstein summation convention is used.

where the values $a^\rho=0$, $\rho=1,\dots,r$, give the identity transformation, i.e. if $\delta A=0$ for the infinitesimal transformations

$$\delta x^\mu=X^\mu_\rho(x,\varphi)\,a^\rho,\quad |a^\rho|\ll 1,$$

$$\delta\varphi^i=Z^i_\rho(x,\varphi)\,a^\rho,$$

then there exists r independent conserved currents

$$\begin{aligned}j^\mu_\rho&=T^\mu_\nu X^\nu_\rho-\frac{\partial L}{\partial(\chi_\mu\varphi^i)}Z^i_\rho,\quad\rho=1,\dots,r,\\T^\mu_\nu&=\frac{\partial L}{\partial(\partial_\mu\varphi^i)}\partial_\nu\varphi^i-\delta^\mu_\nu L,\end{aligned}\tag{2}$$

for the solution $\varphi^i(x)$ of the equations $E_i(\varphi)=0$.

Examples:

i. Translations:

$$x^\mu\to x^\mu+a^\mu,\quad\delta\varphi^i=0;\quad j^\mu_\nu=T^\mu_\nu,\quad\mu,\nu=1,\dots,m.$$

ii. Internal symmetries:

$$\delta x^\mu=0,\quad\hat{\varphi}^i(x)=C^i_j(a^1,\dots,a^r)\,\varphi^j(x),$$

$$j^\mu_\rho=-\frac{\partial L}{\partial(\partial_\mu\varphi^i)}Z^i_\rho,\quad\rho=1,\dots,r.$$

II. If the action integral is invariant under an «infinite-dimensional» (gauge) group the elements of which depend on r smooth functions $g^\rho(x)$, $\rho=1,\dots,r$ and their derivatives up to order s_ρ such that

$$\delta\varphi^i=\sum_{\rho=1}^{r}\sum_{\sigma_1,\dots,\sigma_m=0}^{\sigma_1+\dots+\sigma_m=s_\rho}a^i(x,\varphi,\partial\varphi)_{\rho;\sigma_1\dots\sigma_m}\frac{\partial^{\sigma_1+\dots+\sigma_m}}{\partial(x^1)^{\sigma_1}\dots\partial(x^m)^{\sigma_m}}g^\rho(x),$$

then there exists r identities

$$\sum_{\sigma_1,\dots,\sigma_m=0}^{\sigma_1+\dots+\sigma_m=s_\rho}(-1)^{\sigma_1+\dots+\sigma_m}\frac{\partial^{\sigma_1+\dots+\sigma_m}}{\partial(x^1)^{\sigma_1}\dots\partial(x^m)^{\sigma_m}}\left(a^i(x,\varphi,\partial\varphi)_{\rho;\sigma_1\dots\sigma_m}E_i(\varphi)\right)=0,$$

$$\rho=1,\dots,r$$

between the n Euler-Lagrange expressions $E_i(\varphi)$. The proof uses partial integration and the fact that one can choose $g^\rho(x)=0$ and $(\partial^{\sigma_1+\dots+\sigma_m}/\partial(x^1)^{\sigma_1}\dots\partial(x^m)^{\sigma_m})\, g^\rho=0$ on the boundary ∂G.

Examples:

i. Electrodynamics: Here we have

$$E_\mu(A)=\partial^\nu F_{\nu\mu},\quad F_{\nu\mu}=\partial_\nu A_\mu-\partial_\mu A_\nu\,,$$

and the invariance of the action integral $-\frac{1}{4}\int d^4x\; F_{\mu\nu}F^{\mu\nu}$ under the gauge transformation $\delta A^\mu=\partial^\mu g(x)$ implies $\partial^\mu E_\mu(A)=0$, which is, of course, a consequence of the antisymmetry $F_{\mu\nu}=-F_{\nu\mu}$.

ii. General Relativity: In this case we have

$$E_{\mu\nu}(g)=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=:G_{\mu\nu}\,,$$

where $R_{\mu\nu}$ is the Ricci tensor and $R=g^{\mu\nu}R_{\mu\nu}$. The invariance of the action integral

$\int d^4x\sqrt{-g}\;R$ under infinitesimal coordinate transformations

$$\delta x^\mu=h^\mu(x),\quad \delta g_{\mu\nu}=D_\mu h_\nu+D_\nu h_\mu\,,$$

where D_μ is the covariant derivative, yields the 4 (contracted) Bianchi identities

$$D^\mu E_{\mu\nu}(g)=0\,,$$

which were discovered by Hilbert and discussed by him in his first communication[6] on general relativity.

3. Invariant theory

> The theory of invariants came into existence about the middle of the nineteenth century somewhat like Minerva: a grown-up virgin, mailed in the shining armor of algebra, she sprang forth from Cayley's Jovian head.
> Hermann Weyl[7] (1885-1955)

Emmy Noether[8] (1882-1935) got her Ph.D.-degree in mathematics in 1907

6 HILBERT 1915.

7 WEYL 1939. As to Weyl see: CHEVALLEY and WEIL 1957.

8 There exists a considerable amount of literature on Emmy Noether's life and work: WEYL 1935; VAN DER WAERDEN 1935; ALEXANDROFF 1936; DICK 1981; BREWER and SMITH 1981 (in this volume the first chapter by Clark Kimberling is of special interest). I have seen the announcement —but not the book itself— of the proceedings: SRINIVASAN *et al.*, 1983.

from the University of Erlangen. Her thesis adviser was Paul Gordan[9] (1837-1912), a colleague of her father, the mathematician Max Noether[10] (1844-1921). Both had been students and collaborators of Alfred Clebsch[11] (1833-1872). Whereas Emmy's father had worked mainly in algebraic geometry,[12] Gordan was a specialist in algebraic invariant theory.[13]

Algebraic invariant thoery had been created by Arthur Cayley[14] (1821-1895) in the year 1845 and worked out in close collaboration with James Joseph Sylvester[15] (1814-1897). Both were barristers in London during those fruitful years, earning their living by practicing law!

Algebraic invariant theory deals with multilinear forms, e.g.

$$F^{(m,p)}(x_1,\dots,x_n;\,a) \tag{3}$$

$$=\sum_{i_1,\dots,i_m=1}^{n} a_{i_1\dots i_m}(x_{i_1})^{\alpha_{i_1}}\dots(x_{i_m})^{\alpha_{i_m}},\quad \alpha_{i_1}+\dots+\alpha_{i_m}=p\,.$$

If one passes from the variables x_i to the variables y_i by a linear transformation $x_i=c_{ij}y_j$, $|(c_{ij})|\neq 0$, and inserts these expressions into the form (3), then a new form of the same type results:

$$G^{(m,p)}(y;\,b)=F^{(m,p)}[x(y;c);\,a]\,,$$

where the coefficients $b^{i_1\dots i_m}$ are functions of the coefficients $a^{i_1\dots i_m}$ and the matrix elements c_{ij}. The main question in algebraic invariant theory then is: Which algebraic functions $I(a)$ of the coefficients $a^{i_1\dots i_m}$ are invariant under linear transformations, such that $I(b)=|(c_{ij})|^g\,I(a)$, where g is some rational number.

Example:

$$F^{(2,2)}(x_1,x_2;a)=a_{11}x_1^2+2a_{12}x_1x_2+a_{22}x_2^2\,,$$

$$I(a)=a_{11}a_{22}-(a_{12})^2,\quad g=1\,.$$

I remark in passing that the notions «invariant», «covariant», «contravariant», «cogredient», «contragredient» and others are all due to Sylvester,[16] who led a

9 As to Gordan see NOETHER Max 1914.

10 As to Max Noether as Emmy's father and as mathematician see the literature quoted in footnote 8. Concerning his mathematical work see: BRILL 1923; CASTELNUOVO, ENRIQUES and SEVERI 1925.

11 As to Clebsch see: BRILL, GORDAN, KLEIN, LÜROTH, MAYER, NOETHER and VON DER MÜHLL 1874.

12 As to the history of algebraic geometry and Max Noether's contribution in this field see: DIEUDONNÉ 1972.

13 The following literature deals with invariant theory and its history: MEYER 1898; WEITZENBÖCK 1923; WEITZENBÖCK 1927; SCHUR 1928; WEYL 1946; ch.II; FISCHER 1966; DIEUDONNÉ and CARRELL 1970.

14 CAYLEY 1845 and 1846a. A slightly extended French version of these two papers appeared in CAYLEY 1846b. As to Cayley see: FORSYTH 1895; NOETHER, M. 1895.

15 As to Sylvester see: BAKER 1912; NOETHER, M. 1898.

16 SYLVESTER 1851 (introduces the notions «covariant», «contravariant» and «invariant»); SYLVESTER 1852 (introduces the notions «cogredient» and «contragredient»). In the following I quote according to the *Collected Papers* of Sylvester.

rather restless life, wrote poetry which he read in public recitals and gave many mathematical concepts their lasting names.[17] In mathematics he had the reputation that in his creative periods when he was flooded with new ideas he wrote them down for publication almost instantly, without caring too much about the details of the proofs![18]

Sylvester also introduced[19] the notion of «infinitesimal transformations», referring to the works of Cayley, Aronhold[20] (1819-1884) and Eisenstein[21] (1823-1852). Sylvester was so enthusiastic about the introduction of this concept that he made a footnote in which he said:[22] «... and I take this opportunity of adding that I shall feel grateful for the communication of any ideas and suggestions relating to this new Calculus from any quarter and in any of the ordinary mediums of language—French, Italian, Latin or German, provided that it be in the Latin character.»

In this *Thèse* from 1878 the French mathematician George-Henri Halphen[23] (1844-1889) introduced and analyzed[24] the concept of differential invariants («invariants differéntiels»). The concept had been implicitly dealt with earlier[25] by Lie and when Lie learnt about Halphen's work, he had one of his priority worries because the thought that Halphen did not give him proper credit.[26] Halphen's work stimulated Lie to write several important papers[27] on differential invariants!

Let $F_\alpha(x_1, ..., x_n)$, $\alpha = 1, ..., a$, be some smooth functions and $x_i = f_i(y_1, ..., y_n)$, $i = 1,..., n$, $|(\partial x/\partial y)| \neq 0$ a regular transformation.

Define $\hat{F}_\alpha(y) = F_\alpha(f(y))$. If

$$\begin{aligned} &J(x,\ F(x),\ \partial F/\partial x,\ \partial^2 F/\partial x^2, ..., dx) \\ &= J(y,\ \hat{F}(y),\ \partial\hat{F}/\partial y,\ \partial^2\hat{F}/\partial y^2,\ ...,\ dy), \end{aligned} \tag{4}$$

then the quantity $J(y, F, \partial F/\partial y, ...)$ is called a differential invariant. Notice that the

17 See footnote 15.
18 See the remarks by Noether at the end of his obituary (NOETHER, M. 1898), and the letter by G. Salmon to Sylvester as quoted by BAKER 1912, xxvi.
19 SYLVESTER 1852, pp. 326, 351sqq. Sylvester's notion of «infinitesimal variations» is, however, not so new as he or others working in invariant theory thought (see, e.g. CAYLEY *Papers*, vol. II (1889), 601/601: notes by Cayley himself.) It certainly was used before, e.g., by Lagrange, Jacobi and Hamilton, see my section 6 below.
20 SYLVESTER 1852, 351/352. It is not clear from Sylvester's wording to which work of Aronhold he is referring. Siegfried Heinrich Aronhold was well-known in the 19th century for his work on algebraic invariant theory: *Allgemeine Deutsche Biographie*, Bd. 46 (Leipzig 1902) 58-59.
21 I could not find any paper by Eisenstein which would correspond to Sylvester's remark. As to the work and the short life of the mathematical prodigy Eisenstein see: EISENSTEIN *Papers*. The second volume contains several articles on the life of Eisenstein.
22 SYLVESTER 1852, 352.
23 As to Halphen see the notices by E. Picard and H. Poincaré at the beginning of the first volume of HALPHEN *Papers*, viii-xliii.
24 G.-H. Halphen, *Sur les Invariants Différentiels;* HALPHEN *Papers* vol. II (1918), 197-352. This volume contains other papers by Halphen on the same subject.
25 LIE 1872a and 1872b, 1874b, 1874c, and 1875.
26 See the correspondence between Lie and A. Mayer and F. Klein: LIE *Papers*, vol. VI (1927), 777-793.
27 LIE 1884, 1885.

assumptions of Noether's theorems are fullfilled if $L(x,\varphi,\partial\varphi)dx^1 ... dx^m$ is a differential invariant under infinitesimal transformations.

In his famous work[28] on the 3-body problem Henri Poincaré[29] (1854-1912) introduced the notion of «integral invariants» which is very closely related to that of differential invariants:[30] The quantity $I(x,F,\partial F/\partial x...)$ is called an integral invariant if

$$\int_G I[x,\ F(x),\ \partial F/\partial x,\ ...]\ dx_1 ... dx_n = \int_{\hat{G}} I[y,\ \hat{F}(y),\ \partial\hat{F}/\partial y ...]\ dy_1 ... dy_n,$$

where y and $F(y)$ have the same meaning as above. In Poincaré's case the mapping $x \rightarrow y$ is given by the flow $(q(t_1), p(t_1)) \rightarrow (q(t_2), p(t_2))$ in phase space.

4. Riemannian geometry, calculus of variations and Einstein's theory of gravitation

> Die Frage über die Gültigkeit der Voraussetzungen der Geometrie im Unendlichkleinen hängt zusammen mit der Frage nach dem inneren Grunde der Maßverhältnisse des Raumes... Die Entscheidung dieser Fragen kann nur gefunden werden indem man von der bisherigen durch die Erfahrung bewährten Auffassung der Erscheinungen, wozu Newton den Grund gelegt, ausgeht und diese durch Tatsachen, die sich aus ihr nicht erklären lassen, getrieben allmählich umarbeitet;... Es führt dies hinüber in das Gebiet einer andern Wissenschaft, in das Gebiet der Physik....
> **Bernhard** Riemann[31] (1826-1866)

The genesis of Emmy Noether's paper is very closely related to David Hilbert's work on Albert Einstein's theory of gravitation.[32] In 1915 Hilbert[33] (1862-1943) had become interested in this theory and he derived the final equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa T_{\mu\nu}$$

simultaneously with Einstein[34] (1879-1955) himself!

Let me briefly recall a few dates:

28 POINCARÉ 1890, ch. II, paragraph 6; 1892, t. III, ch. 22.
29 As to Poincaré see: LEBON 1912. *Acta Mathematica 38* (1921): «Henri Poincaré in Memoriam». BROWDER 1983.
30 LIE 1897; WEITZENBÖCK 1923.
31 The quotation is from the last paragraph of Riemann's famous inaugural lecture, RIEMANN 1854. Translation: «The question of the validity of the geometrical assumptions about the infinitely small distances is related to the question concerning the deeper reasons for the geometry of space... A decision on these questions can only be found if one starts from the present, empirically tested concepts of the phenomena, for which Newton laid the foundations, and compelled by facts which cannot be explained by them, gradually modifies these concepts;... This leads us into the field of another science, into that of physics...» As to Riemann see: RIEMANN *Papers* 541-558; KLEIN 1894 and COURANT 1926.
32 HILBERT 1915, 1917a and 1924.
33 As to Hilbert see: «David Hilbert zur Feier seines sechzigsten Geburtstages» (with contributions from O. Blumenthal, O. Toeplitz, M. Dehn, R. Courant, M. Born, P. Bernays and K. Siegel) in: *Die Naturwissenschaften 10* (1922) 65-103. Otto Blumenthal, Lebensgeschichte, in: HILBERT *Papers*, vol. III, 388-429. WEYL 1944. REID 1970.
34 On Einstein and especially on his relationship to Hilbert see the beautiful Einstein-biography by PAIS 1982 ch. IV.

On Nov. 11, 1915, Einstein presented[35] at a session of the Prussian Academy of Sciences in Berlin his newest version of the gravitational field equations:

$$R_{\mu\nu} = -\kappa T_{\mu\nu},$$

where he had to make the consistency assumptions $\sqrt{-g}=1, g=|(g_{\mu\nu})|, T^{\mu}_{\mu}=0$. Immediately afterwards Einstein realized that he could get rid of these constraints if he replaced the above field equations by the following ones:

$$R_{\mu\nu} = -\kappa\,(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\, T^{\rho}_{\rho}).$$

He communicated[36] this final version of this theory during a session of the Academy on Nov. 25, 1915.

On Nov. 20, 1915, Hilbert presented[37] his derivation of the same equations at a session of the «Royal Society» in Göttingen. He derived the field equations from the action integral

$$\int d^4x \sqrt{-g}\,(R + L_{\text{el.magn.}})$$

and he noticed that 4 of the Euler-Lagrange equations were a consequence of the others, due to the fact that the action integral was invariant under arbitrary coordinate transformations.

At the end of his communication Hilbert praises the «axiomatic method», he had used, «which here, as we see, employs the most powerful instruments of analysis, namely the calculus of variations and invariant theory».

Here we get back to Emmy Noether:[38] Her work in Erlangen on invariant theory[39] had attracted the interest of Hilbert and Klein and they invited her to Göttingen. She went there in the Spring of 1915. In November of the same year she wrote to the Erlanger mathematician Ernst Fischer[40] (1875-1954): «Invariantentheorie ist hier Trumpf; sogar der Physiker Hertz studiert Gordan-Kerschensteiner; Hilbert will nächste Woche über seine Einsteinschen Differential-invarianten vortragen, und da müssen die Göttinger doch etwas können».[41]

During the winter term 1916/17 she gave lectures on invariant theory and she worked on invariants of differential forms

$$f(x,dx) = \Sigma\, g_{ij\cdots s}(x)\, dx^i dx^j \ldots dx^s$$

35 EINSTEIN 1915a.
36 EINSTEIN 1915b.
37 HILBERT 1915. In this printed version of his communication Hilbert quotes the one by Einstein from Nov. 25 which was printed on Dec. 2.
38 In the following I rely on the literature quoted in footnote 8.
39 As to Emmy Noether's early work on invariant theory see: NOETHER *Papers*, articles 1-4, 7 and 8.
40 As to Fischer see: M. Pinl, «Ernst Sigismund Fischer»; in: *Neue Deutsche Biographie*, Bd. 5 (Duncker und Humboldt, Berlin 1961) 183.
41 BREWER and SMITH 1981, p. 12. Here she is referring to GORDAN 1885.

in close contact with Hilbert and Klein, as can be seen from the exchange of letters[42] between the two, in which both refer to the help they received from «Fräulein Noether».

The first paper[43] «Invarianten beliebiger Differentialausdrücke», which contained a part of her work, was communicated by Klein at a session of Göttingen's Royal Society in Jan. 1918.

In this paper E. Noether gave a general procedure for calculating all differential invariants of a differential form, using ideas of Riemann,[44] Christoffel[45] (1829-1900) and Lipschitz[46] (1832-1903) and relating the differential invariants to the different orders of «variations» in the calculus of variations:

Example: From the «binary» form

$$f(x,dx) = g_{ij}(x)\; dx^i\, dx^j$$

one can calculate

$$f_{\delta} := \frac{\partial f}{\partial(dx^j)}\,\delta x^j = 2g_{ij}\, dx^i\, \delta x^j,$$

$$\delta f = (\delta g_{ij})\, dx^i\, dx^j + 2g_{ij}\,(\delta dx^i)\, dx^j\,,$$

$$df_{\delta} = 2\,(dg_{ij})\, dx^i\, \delta x^j + 2g_{ij}\,(d^2x^i)\,\delta x^j + 2g_{ij}\, dx^i (d\delta x^j).$$

The difference

$$\delta f - df_{\delta} = -2\, L_j\, \delta x^j\,,$$

$$L_j(x) = g_{jk}\, d^2x^k + \Gamma_{j,kl}\, dx^k dx^l = E_j(x)\, dt^2$$

of the 2 differential invariants δf and df_{δ} is again a differential invariant and the last equation shows that the Euler-Lagrange expressions $E_j(x)$ of the variational problem

$$\delta \int dt\, [f(x,dx/dt)]^{1/2} = 0$$

42 KLEIN 1917.
43 NOETHER, E. 1918a.
44 RIEMANN 1861. This important paper by Riemann, which was first published in 1876 in the first edition of Riemann's collected papers, contains the essential ideas for the mathematical proofs of the assertions he made in his inaugural lecture, RIEMANN 1854. Riemann submitted this paper in 1861 to the Académie des Sciences in Paris which had invited for prize essays. Riemann did not receive the prize, however, because his proofs were considered incomplete! When Riemann's paper was finally published, others —BELTRAMI 1868; Lipschitz and Christoffel (see the next two references)— in the meantime had provided the proofs, too.
45 CHRISTOFFEL 1869. As to Christoffel see: BUTZER and FÉHER 1981.
46 LIPSCHITZ 1869, 1870a, 1870b, 1872, 1877. As to Lipschitz see: KORTUM 1906.

for the geodesics form a covariant vector. The details, including the higher order differential invariants (curvature etc.) for this example were worked out[47] by H. A. Hermann Vermeil (1889-1959), according to Noether's ideas. When Einstein saw Noether's paper, he was quite impressed as can be seen from the following letter to Hilbert[48] from May 24, 1918: «Gestern erhielt ich von Frl. Noether eine sehr interessante Arbeit über Invariantenbildung. Es imponiert mir, daß man diese Dinge von so allgemeinem Standpunkt übersehen kann. Es hätte den Göttinger Feldgrauen nichts geschadet, wenn sie zu Frl. Noether in die Schule geschickt worden wären. Sie scheint ihr Handwerk zu verstehen!»

Emmy Noether's two papers from January and July 1918 are closely related to the interests of Felix Klein at that time, which can be seen from Klein's communications[49] to the Royal Society in Göttingen and from the second volume[50] of his lectures on the development of mathematics in the 19th century, which he held during the years 1915-17.

Klein's interest in the subject was kindled by the relations he saw between ideas in special and general relativity and his «Erlanger Programm» on transformation groups and their invariants (see below) and by his deep admiration for Riemann,[51] whom he saw so surprisingly justified by Einstein's theory of gravitation. Concerning this theory Klein was worried[49] that the energy-momentum continuity equations followed from the identities $D^{\mu}G_{\mu\nu}=0$ and, unlike in mechanics and electrodynamics, were not a consequence of the equations, of motion or field equations (the analogy to charge conservation in electrodynamics and its connection with gauge invariance he apparently did not see! This was first pointed out by Erich Bessel-Hagen[52] (1898-1946)).

The connection between the 10 classical conservation laws (energy, momenta, angular momenta and uniform centre of mass motion) and the corresponding space-time symmetries (time and space translations, rotations and Galileo or special Lorentz transformations) had interested Klein for several years:

In 1911 the mathematician Gustav Herglotz[53] (1881-1953) adapted non-relativistic continuum mechanics to the framework of special relativity. In that paper Herglotz derived the 10 classical conservation laws from the invariance of the action integral under the 10-parameter inhomogeneous Lorentz group by using essentially the same procedure as E. Noether did several years later, by means of the relation (1) above. Herglotz's derivation of the classical conservation laws for a field theory (continuum mechanics) from the invariance of its action integral under transformation groups has to be counted among the most important contributions in this field among those preceding Noether's!

47 VERMEIL 1918.
48 BREWER and SMITH 1981, 13, with an English translation on p. 46.
49 KLEIN 1917, 1918a, 1918b.
50 KLEIN 1927, ch. 3.
51 KLEIN 1894, 1926.
52 BESSEL-HAGEN 1921. As to Bessel-Hagen see: POGGENDORFF *Biogr.* 1956, Bd. VIIa, Teil 1 (A-E), p. 166.
53 HERGLOTZ 1911, paragraph 9. As to Herglotz see: TIETZE 1954.

Herglotz was well-known to Klein: After receiving his Ph. D. in Munich as a student of the astronomer Seeliger, he had been in Göttingen from 1903 till 1908, where, at the suggestion of Klein, he had become *Privatdozent* and *Ausserordentlicher* Professor. When Klein saw Herglotz's paper from 1911, he realized that the connection between symmetry properties of a system and its conservation laws discussed there was related to Lie's work on group theory applied to differential equations (see below). So he asked his former student and Lie's collaborator for many years, Friedrich Engel[54] (1861-1941), to derive the 10 classical conservation laws for an n-body problem with 2-body potential forces from the invariance under the 10-parameter Galileo-group in the framework of Lie's theory.

Engel proved[55] the following:

Define the variable p by $H+p=$ const., where H is the Hamilton function of the system, which does not depend on the time t explicitly. Suppose $F(t,q^1,\dots,q^{3n};p,p_1,\dots,p_{3n})$ is the generating function of an infinitesimal canonical transformation, i.e.,

$$\delta q^i = F_{p_i}\,\delta a, \qquad \delta t = F_p\,\delta a,$$

$$\delta p_i = -F_{q^i}\delta a, \qquad \delta p = -F_t\,\delta a$$

(the subscript means partial derivative with respect to the corresponding variable), such that the 1-form $p_j dq^j + p dt$ remains invariant under the infinitesimal transformation up to the total derivative of a function, then F is a consant of motion! Constructing the corresponding generating functions for space and time translations etc. Engel derived the 10 classical conservation laws. Since $p_j dq^j + p dt = p_j dq^j - H dt + \text{const.}\ dt = (L+\text{const.})\ dt$, this result is equivalent to that of E. Noether, applied to mechanics.

In this paper Engel derived the Lie algebra of the 10-parameter Galileo group, too! Klein was not yet satisfied. He asked Engel whether the invariance of the *equations of motion* for the gravitational n-body problem under the scale transformation $x_i \to \lambda^2 x_i$, $t \to \lambda^3 t$ would lead to a reduction of the degree of integrations necessary in order to obtain the solutions. Engel's answer was[56] that it would not, a result which had already been obtained by Poincaré[57] in 1890, whom Engel, however, does not mention. Notice that the above scale transformation does not leave the action integral invariant.

At this point the question arises, when it was realized that the form invariance of the action integral implies the corresponding invariance of the equations of motion or field equations. As far as I could find out, this insight evolved in connection with discussions of the Lorentz invariance of the action integral for electrodynamics after Einstein in 1905 had introduced[58] his «principle of relativity». Ideas similar to that

54 As to Engel see: ENGEL 1938. FABER and ULLRICH 1945.
55 ENGEL 1916. See also: ENGEL and FABER 1932, ch. 10.
56 ENGEL 1917.
57 POINCARÉ 1890, p. 51/52.
58 EINSTEIN 1905; On the historical impact of this paper see: MILLER 1981 and PAIS 1982, ch. III.

of a least action for mechanical systems[59] were first applied to electrodynamics (and reversible thermodynamics) by Hermann L. F. von Helmholtz[60] (1821-1894) in 1892. As Helmholtz wanted to include into his action the dielectric and magnetic properties of matter, his Lagrangian is rather complicated. In modern notation his action for the «free ether» amounts to

$$\int dt\, d^3\mathbf{x}\, [\tfrac{1}{2}\mathbf{E}^2 + \tfrac{1}{2}(\mathrm{curl}\mathbf{A})^2 - (\mathbf{j} + \partial_t\mathbf{E})\cdot\mathbf{A} + \ldots]\,,$$

with the vector potential **A** and the electric field **E** as the quantities to be varied independently.

In 1900 Joseph Larmor[61] (1857-1942) derived several of Maxwell's equations and the Lorentz-force from the action integral

$$\int dt\, d^3\mathbf{x}\, [\tfrac{1}{2}(\mathrm{curl}\mathbf{A})^2 - \tfrac{1}{2}\mathbf{E}^2 + \varphi(\mathrm{div}\mathbf{E} - \rho)] + \int dt\, \tfrac{1}{2}m\dot{\mathbf{x}}^2\,,$$

assuming the validity of the equation $\mathrm{curl}\mathbf{H} = \mathbf{j} + \partial_t\mathbf{E}$ and using the scalar potential φ as a Lagrangian multiplier. Notice that

$$\mathbf{A}\ \mathrm{curl}\mathbf{H} = (\mathrm{curl}\mathbf{A})^2 + \mathrm{div}(\mathbf{H}\times\mathbf{A}),\ \mathbf{H} = \mathrm{curl}\mathbf{A}.$$

Very similar considerations are contained in Poincaré's lectures on electricity and optics from 1899, published[62] in 1901.

In 1903 Karl Schwarzschild[63] (1873-1916) gave the electromagnetic action its modern version: He derived Maxwell's eqs. and the Lorentz force from the action integral

$$\int dt\, d^3\mathbf{x}\, [\tfrac{1}{2}(\mathrm{grad}\varphi + \partial_t\mathbf{A})^2 - \tfrac{1}{2}(\mathrm{curl}\mathbf{A})^2 - \rho\varphi + \mathbf{j}\cdot\mathbf{A}] + \int dt \tfrac{1}{2}m\dot{\mathbf{x}}^2\,,$$

with the potentials φ and **A** as the independent field variables. In 1904 the different existing versions of electrodynamic action integrals were summarized by Hendrik

59 There exists a lot of literature on the history of the principle of least action. A selection is: MAYER 1886, HELMHOLTZ 1887, HÖLDER 1896, JOURDAIN 1908b, 1914, 1908a, 1913, KNESER 1928, and BRUNET 1938.

60 HELMHOLTZ 1892. Helmholtz's work was immediately (1893) incorporated by Boltzmann into his lectures on Maxwell's theory of electromagnetism, see: BOLTZMANN 1982, Part II, 1st lecture. Maxwell himself already pointed out that Lagrange's form of the mechanical equations of motion could be useful for the equations of electromagnetism, too; see: MAXWELL 1873, vol. II, paragraphs 553-584. See also KAISER 1982, esp. pp. 20-29*. As to H. v. Helmholtz see: KÖNIGSBERGER 1902. «Dem Andenken an Helmholtz» (with contributions of J. von Kries, W. Wien, W. Nernst, A. Riehl and E. Goldstein); in: *Die Naturwissenschaften 9* (1921) 673-708.

61 LARMOR 1900, ch.VI. Larmor's variations are somewhat obscure! As to Larmor see: EDDINGTON 1942.

62 POINCARÉ 1901a, 3e partie, ch. III.

63 SCHWARZSCHILD 1903. As to Schwarzschild see: SOMMERFELD 1916.

Antoon Lorentz[64] (1853-1928) in his two well-known *Encyklopädie*-articles.[65] Lorentz himself has derived the electrodynamical equations in 1892 from a generalization of d'Alembert's principle.[66]

In his famous relativity paper frm 1905 Poincaré derived[67] Maxwell's equations from the action

$$\int dt\, d^3\mathbf{x}\, [\tfrac{1}{2}\mathbf{E}^2 + \tfrac{1}{2}(\mathrm{curl}\mathbf{A})^2 - (\mathbf{j} + \partial_t\mathbf{E})\cdot\mathbf{A} - \varphi(\mathrm{div}\mathbf{E} - \rho)]\,,$$

and he observed that the first 3 terms in this action are reduced to the Lorentz invariant expression $\frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2)$ when the field equations $\mathrm{curl}\mathbf{H} = \mathbf{j} + \partial_t\mathbf{E}$ are used in the same manner as indicated above. In 1909 Max Born[68] (1882-1970) observed that the Schwarzschild action could be rewritten as a Lorentz invariant within Hermann Minkowski's[69] (1864-1909) space-time framework. The first who spelt out explicitly that the covariance of the field equations would be guaranteed if the Lagrangian were constructed in terms of the Lorentz invariants $\mathbf{E}^2 - \mathbf{B}^2$, $\mathbf{E}\cdot\mathbf{B}$, $\rho\varphi - \mathbf{j}\cdot\mathbf{A}$ etc., was Gustav Mie[70] (1868-1957). Stimulated by Mie's observation, Hilbert constructed his action integral for gravitation coupled to electromagnetism.[71]

We have seen Felix Klein in the background of several papers dealing with the connection between symmetry groups and conservation laws. Two years after E. Noether's paper from July 1918 he asked E. Bessel-Hagen to apply her results to Galileo's invariance of mechanics and conformal invariance of electrodynamics.[72]

Klein himself later said[73] that after his nervous breakdown in 1882, following his intense publication competition with Poincaré on automorphic functions:[74] «...I introduced a method of scientific work which I employed from that time on: I limited myself to ideas and guide-lines and left the carrying out of the details and further development to younger people, who stood by to help me».

That others did not always see this attitude in the same light is reflected in a somewhat nasty joke about Klein[75] which was told in Göttingen:

The set of mathematicians in Göttingen consists of two disjoint sets: one which contains people who work on problems of Klein's choice but not of their own: the second set contains those who work on problems of their own choice, which is,

64 As to Lorentz see: PLANCK 1928; BORN 1928; EINSTEIN 1973, 70-76; DEHAAS-LORENTZ 1957.

65 LORENTZ 1904.

66 LORENTZ 1892.

67 POINCARÉ 1906.

68 BORN 1909. As to Born see: HEISENBERG 1970; BORN 1968, 1969.

69 MINKOWSKI 1908a. As to Minkowski see: HILBERT 1910; BORN 1959.

70 MIE 1912. As to Mie see: KAST 1957; POGGENDORFF *Biogr.* 1958, VIIa, Teil 3 (L-R) 304.

71 HILBERT 1915, 1917a and 1924.

72 BESSEL-HAGEN 1921.

73 KLEIN 1923.

74 Klein's version of the story and his correspondence with Poincaré can be found in: KLEIN *Papers*, vol. III, pp. 577-621; the correspondence is published in *Acta Math. 39* (1923) 94-132 too. See also KLEIN 1926, ch. III.

75 REID 1970, pp. 88/89.

however, not Klein's. Since Klein does not belong to either of these two sets, he is not a mathematician!

5. Lie's Theory on the Integration of Differential Equations by means of their Symmetry Groups and Klein's «Erlanger Programm»

> De là par exemple est née une théorie générale d'intégration pour les sytèmes d'équations différentielles dont la solution la plus générale s'exprime en fonction d'une solution particulière par des formules qui définissent un groupe fini et continu; cette théorie a une analogie frappante avec celle de *Galois*. Dans chaque cas particulier, en effet, la difficulté du problème d'intégration dépend uniquement de la structure du groupe continu correspondant. Par suite la recherche de la structure de tous les groupes simples a une importance capitale.
> Sophus Lie[76]

For the winter term 1869/70 Felix Klein[77] (1849-1925), 20 years old, went from Göttingen to Berlin, in order to pursue his mathematical studies at the University of Berlin, where Karl Weierstrass[78] (1815-1897), Ernst Eduard Kummer[79] (1810-1893) and Leopold Kronecker[80] (1823-1891) were the leading mathematicians. In Berlin Klein met the Norwegian Sophus Lie[81] (1842-1899), who was there on a Norwegian fellowship. Klein and Lie became friends and this —not always harmonious— friendship turned out to be decisive for the historical development of group theory and its applications!

Klein, who in 1865 at the age of 16 had become Julius Plücker's[82] (1801-1868) assistant in Bonn, came into contact with Clebsch[83] when, after Plücker's death in 1868, he was asked to edit a part of Plücker's latest geometrical work, another part being edited by Clebsch.

In 1868 Clebsch had moved from Giessen to Göttingen, so Klein, after his Ph.D. examination at Bonn in December 1868, went to Göttingen in order to join the stimulating group of young mathematicians around Clebsch.[84]

Lie, who had discovered his interest in mathematics rather late, was strongly interested in geometrical problems[85] and considered Plücker as one of his main

76 Lie 1895.
77 As to Klein see: «Felix Klein zur Feier seines siebzigsten Geburtstages» (with contributions by R. Fricke, A. Voss, W. Wirtinger, A. Schoenflies, C. Carathéodory, A. Sommerfeld, H. E. Timmerding and L. Prandtl); in: *Die Naturwissenschaften 7* (1919) 273-317. Courant 1925; Weyl 1930; Behnke 1960. Of considerable interest are Klein's introductory notes, footnotes and comments in the 3 volumes of Klein *Papers*, published in 1921, 1922 and 1923. A lot about Klein can also be found in the biographies of Hilbert and Courant, Reid 1970 and 1976.
78 As to Weierstrass see: *Acta Mathematica 39* (1923) (volume in memory of K. Weierstrass, H. Poincaré and S. Kowalewsky); Behnke and Kopfermann 1966; Biermann 1966.
79 As to Kummer see: Kummer *Papers*, vol. I: contains several obituaries and other articles on Kummer's work and life.
80 As to Kronecker see: Frobenius 1893, Weber 1893, and Kneser 1925.
81 As to S. Lie see: Engel 1899a, 1899b; and Noether 1900. Of considerable interest are also the many letters and remarks contained in the editorial comments and notes by F. Engel and P. Heegaard at the end of each of the seven volumes of Lie *Papers*.
82 As to Plücker see: Clebsch 1895, and Klein 1926, 119-126.
83 As to the following see the literature quoted in footnote 77.
84 See Brill, Gordan,... 1874, and Klein 1926, 296-298.
85 See footnote 81.

teachers, though he had never met him! Thus, Klein, and Lie had strong common scientific interests, which, however, did not fit so well into the style of mathematics they found in Berlin.

They decided, therefore, to go to Paris. This they did in the spring of 1870. Even though their stay was cut short by the outbreak of the French-German war in July 1870, it had a strong decisive influence on their future scientific work:

In Paris Klein and Lie had their encounter with group theory in the person of Camille Jordan[86] (1838-1922). Between 1867 and 1869 Jordan had published several papers[87] on the Euclidean group in space, in which he had given a thorough analysis of its continuous and discrete subgroups. He also employed the concept of infinitesimal transformations. In 1870 Jordan in his almost 700 pages long *Traité des substitutions et des équations algébriques*[88] gave an extensive presentation and analysis of Galois's theory on algebraic equations and their (Galois) groups.

I mentioned already that Klein's and Lie's stay in Paris was cut short in July 1870 by the French-German war, but, as Richard Courant in his memorial address afer Klein's death in 1925 said,[89] «when Klein had to leave Paris after a stay of 2 and 1/2 months because of the outbreak of the war, he carried the philosopher's stone with him: he had grasped the notion of a group most thoroughly, that signpost which forthwith led him with unerring security on his scientific way of life». The same can be said for Lie! When Klein had left France, Lie decided to walk (!) to Italy in order to avoid the war. He came, however, only as far as *Fontainebleau*, where he was arrested as a German spy, because the mathematical manuscripts, written in German, made him suspect! He stayed in prison for 4 weeks till the testimony of Darboux freed him! How Lie enjoyed this time in prison can be seen from a later letter (1877) of his to the mathematician Adolph Mayer[90] (1839-1908), with whom he collaborated on partial differential equations.[91] The following passage from that letter[92] sheds some light on Lie's somewhat strangely conditioned creativity: «... This is actually strange; in the last few years I have always made my discoveries, when I was afflicted by misfortune in some way: In the spring of 1872 I had injured my eye and exactly at that time I discovered the method of integration. In January of 1873 my father dies, and I created group theory. In the spring of 1876 several misfortunes hit my wife's next of kin and at exactly that time I developed my new theories of integration. In January of 1877 I injured my shoulder, so that I could not continue writing as usual and on the same evening I had a good idea about minimal surfaces which, at least, brought a lot of pleasure. I found the basic idea for my paper «Über Komplexe» in an equally

86 As to Jordan see: Lebesgue 1923.
87 Jordan 1867 and 1868.
88 Jordan 1870.
89 See Footnote 77.
90 As to Mayer see: Liebmann 1908.
91 This collaboration did not result in joint papers, but Mayer —similar to Klein— had a strong influence on Lie by urging him to clarify his many ideas and by helping Lie to prepare the final version of his papers for publication, a task which Lie did not like at all! The widespread recognition of Lie's and Mayer's work can be judged, for instance, from the wellknown textbook: Goursat 1921, chs. VIII-XI.
92 Lie *Papers*, vol. III, 691.

strange way, one evening in Paris as I had just fallen asleep. Immediately afterwards I was put into prison for one month in Fontainebleau and I had complete peace and quiet working out that discovery which gave me incomparable pleasure.» (The «methods of integration» Lie mentions here refer to his work[93] on the integration of partial differential equations.)

At the end of 1870 Lie went back to Oslo (then Kristiana), where in the summer of 1871 he took his Ph.D. examination.

In the beginning of the same year Klein became *Privatdozent* in Göttingen, working and teaching in close contact with Clebsch. In the wake of their close collaboration in Berlin and Paris, Klein and Lie wrote several joint papers,[94] the most important of which is the one which appeared in vol. 4 of the *Mathematische Annalen*. Here they consider properties of transformation groups in the plane, discuss their orbits, the sets of points left invariant, their infinitesimal transformations and, in the final paragraph, for which Lie was responsible,[95] they show, by means of a simple example, how the knowledge of a transformation group which leaves a differential equation invariant, can help to integrate that equation.

Though the results of this paper are not overwhelming, its content set the future trend in research for its two authors:

In 1871/72 Lie began to work systematically on the problem of how to use the knowledge that a differential equation or a set of them is invariant under an infinitesimal transformation group, for the integration of those equations.

As Lie stressed several times[96] in later years, this approach was a generalization of the Galois theory[97] for algebraic equations

$$P_n(x) \equiv x^n + a_{n-1} x^{n-1} + \dots + a_0 = 0 ,$$

where the knowledge of the discrete Galois (permutation) group, the elements g of which permute the roots x_i, $i = 1, \dots, n$, of the equation $P_n(x) = 0$ but leave this equation invariant, is of essential importance in finding those roots and determining their properties.

After Lie had started this work on differential equations, he realized that he did not know enough about the structure of continuous groups and so he began to analyze them. This work of Lie is much better known[98] than his work on differential equations.

But it should not be forgotten that Lie's work on group theory was mainly

93 Lie's work on partial differential equations is contained mainly in: LIE *Papers*, vols. III, IV and VI.
94 KLEIN and LIE 1870a, 1870b, and 1871.
95 KLEIN *Papers*, vol. I, p. 456, footnote 28. See also: LIE *Papers*, vol. I, 743-746.
96 I quote only a few examples, of which there are many more: LIE and ENGEL 1888, vol. I, p. III-V and vol. III, p. VI-XXIII. See also LIE 1895.
97 There are, of course, many books on Galois Theory; I myself, as a student, tried to learn the rudiments from the classical textbook: VAN DER WAERDEN 1955.
98 In the meantime there are so many books on Lie algebras and Lie groups and since everybody has his favoured choise I do not quote any of them!

motivated by his interest to provide general methods for solving differential equations!

Lie's papers of differential equations and their symmetry groups are contained in the vols. III-VII of his collected work.[99] It is impossible to summarize the wealth of his ideas here and I have to refer to the literature on them.[100] I will mention one simple example:[101]

Suppose the Pfaffian differential equation

$$X(x,y)\, dy - Y(x,y)\, dx = 0$$

has the integral curves $\omega(x,y) = \text{const.}$, i.e. $\omega(x,y)$ obeys the equation $X\partial_x\omega + Y\partial_y\omega = 0$. Suppose further that the infinitesimal transformation

$$\delta x = \xi(x,y)\,\delta a , \qquad \delta y = \eta(x,y)\,\delta a$$

maps the integral curve $\{(x,y)\}$ onto an integral curve $\{(x+\delta x, y+\delta y)\}$. Because $\omega(x+\delta x, y+\delta y) = \omega + (\xi\partial_x\omega + \eta\partial_y\omega)\,\delta a$, the points $(x+\delta x, y+\delta y)$ describe an integral curve iff $\xi\partial_x\omega + \eta\partial_y\omega = f(\omega)$. Now, if φ is a smooth function, then $\varphi(\omega)$ is a solution of the equation $X\partial_x\varphi + Y\partial_y\varphi = 0$, if $\omega(x,y)$ is a solution. Thus, we can normalize $f(\omega) = 1$.

From the equations

$$X\partial_x\omega + Y\partial_y\omega = 0, \qquad \xi\partial_x\omega + \eta\partial_y\omega = 1$$

we obtain

$$\partial_x\omega = -\frac{Y}{\eta X - \xi Y}, \qquad \partial_y\omega = \frac{X}{\eta X - \xi Y},$$

i.e. $\eta X - \xi Y$ is an integrating factor of the above Pfaffian equation and we can calculate the function $\omega(x,y)$ by the line integral

$$\omega(x,y) = \int \frac{X\,dy - Y\,dx}{\eta X - \xi Y}$$

Lie did not only work on finite-dimensional (r-parameter) continuous groups.

99 See the quotations of the different volumes in the references given above.
100 LIE and SCHEFFERS 1891. One of the first who included Lie's theory on partial differential equations in a textbook was GOURSAT 1921, in the first edition from 1891 (Lie, who was very pleased about this book, wrote a preface to the German translation, which was published in 1893 bei Teubner (Leipzig); see LIE *Papers*, vol. IV, 317-319). VESSIOT 1899; the French mathematician Ernest Vessiot (1865-1952) came to Leipzig in 1888 (LIE *Papers*, vol. V, p. 652) in order to study Lie's work, and later made a number of important contributions to the relationship between group theory and the theory of differential equations himself. VON WEBER 1899. COHEN 1911. ENGEL and FABER 1932. Modern textbooks (articles) with many references are the following: BLUMAN and COLE 1974, OVSIANNIKOV 1982, and WINTERNITZ 1983.
101 LIE 1874a. See also LIE and SCHEFFERS 1891, ch. 6.

His work on partial differential equations led him to investigate infinite-dimensional groups, nowadays called pseudogroups by mathematicians[102] and gauge groups by physicists.[103] It is a part of this work Emmy Noether refers to[104] in connection with her second theorem mentioned above.

In the fall of 1872 Felix Klein, 23 years old, was appointed full professor for mathematics at the University of Erlangen. During Klein's last month in Göttingen and his first month in Erlangen Lie joined him, in order to discuss their respective work on group theory.

In Erlangen each new professor had to present an outline («Programm») of his future research to the faculty. In Nov. 1872 Klein presented his famous «Erlanger Programm» on the application of group theory to geometry,[105] as it had emerged from his discussions with Lie.[106] This «manifesto» on the importance of group theory for geometry later was translated into many languages[107] and became very influential![108] It proclaimed for the first time, many ideas concerning transformation groups which nowadays form an essential part of that field:

Klein formulated the general problem he had in mind as follows:[109] «Given are a manifold and a group of transformations of the same; those configurations belonging to that manifold with regard to such properties as are not altered by the transformations of the group should be investigated.» Starting with a «large» («principal») transformation group of a given manifold, one can obtain new geometrical structures by considering the different subgroups of the principal group and by identifying the new geometrical objects which are left invariant by the respective subgroups. Thus, starting with projective geometry and its transformation group, one can characterize affine, metric, conformal etc. geometries by identifying the corresponding subgroups of the general linear transformation group of projective geometry.

After the «Erlanger Programm» was formulated, the scientific paths of Klein and Lie parted.[110] Whereas Lie during the following years worked mainly on differential equations and continuous groups, Klein started to «combine Galois and Riemann», that is to say, he combined the theory of discrete groups with the theory of complex functions. This led to his extensive work on automorphic functions[111] and his fateful scientific encounter with Henri Poincaré, causing his nervous

102 See, for instance, KUMPERA 1975 (with many references).
103 There are many reviews and books on gauge theories, fiber bundles, etc... I mention just one review article which contains many references: EGUCHI, GILKEY and HANSON 1980.
104 LIE 1891.
105 KLEIN 1872.
106 See Klein's notes in: KLEIN *Papers*, vol. I, 411-414.
107 A list of the translations can be found in KLEIN *Papers*, vol. III, Anhang, p. 17. I mention here the —not very smooth— English translation: KLEIN 1893.
108 See, for instance, C. Carathéodory, «Die Bedeutung des Erlanger Programms»; in: *Felix Klein zur Feier....*, compare footnote 77, 297-300.
109 I can mention here only a few of the topics Klein discusses.
110 See Klein as quoted in footnote 106.
111 See footnote 74.

breakdown in 1882 which, as he would later say,[112] destroyed the inner core of his scientific creativity, at the age of 33!

6. Work on relations between conservation laws and symmetry transformations prior to 1918

> In fact, the traditional references to the origin of the fundamental mathematical notions in analytical dynamics are almost always incorrect.
> Aurel Wintner[113] (1903-1958)

I mentioned already the important work of Herglotz and Engel, relating conservation laws of a system to Poincaré —or Galileo— invariance respectively. These papers were referred to by E. Noether. There are older publications on the subject, however, which she does not quote but which are worth mentioning here.

When I looked up the history of the 10 classical conservation laws in mechanics, I was surprised to discover how close their derivations —especially those of momentum and angular momentum conservation— from translation and rotation invariance by Joseph Louis de Lagrange[114] (1736-1813), William Rowan Hamilton[115] (1805-1865) and Carl Gustav Jacob Jacobi[116] (1804-1851) were to the ideas contained in Noether's first theorem: At the center of Lagrange's derivation is his analytical version of the dynamical principle as formulated by Jean Baptiste Lerond D'Alembert[117] (1717-1783): Let $\mathbf{x}_i = (x_i, y_i, z_i)$, $i = 1, \ldots, n$, be the Euclidean position (vector) of a particle, $\mathbf{F}_i(x_1, \ldots, x_n)$ the force acting on it and $\delta\mathbf{x}_i$ an infinitesimal (»virtual») displacement of the position $\mathbf{x}_i$ compatible with the constraints imposed on the system, then the dynamical laws of motion can be derived from the relation

$$\sum_{i=1}^{n} (m_i \ddot{\mathbf{x}}_i - \mathbf{F}_i)\cdot\delta\mathbf{x}_i = 0 \ . \tag{5}$$

Under the assumption that all forces are internal[118] and that they depend only on the relative distances $r_{ik} = [(\mathbf{x}_i - \mathbf{x}_k)^2]^{1/2}$ the sum $\Sigma\ \mathbf{F}_i\cdot\delta\mathbf{x}_i$ becomes a linear combination of the variations

$$\delta r_{ik} = \frac{1}{r_{ik}} (\mathbf{x}_i - \mathbf{x}_k)\cdot\delta(\mathbf{x}_i - \mathbf{x}_k) \ ,$$

112 KLEIN *Papers*, vol. III, p. 585; KLEIN 1926, p. 380.
113 WINTNER 1941, p. 413. As to Wintner see: HARTMAN 1962.
114 LAGRANGE 1853. I shall quote according to LAGRANGE *Papers*, vol. XI. That part which concerns us here is contained in the «Seconde Partie, Sections I-III». As to Lagrange see: BURZIO 1942, and SARTON 1944.
115 HAMILTON 1834. I shall quote according to vol. II of the HAMILTON *Papers*. As to Hamilton see: GRAVES 1882. See also Ch. Graves, «Eloge», in: HAMILTON *Papers*, vol. I, p. ix-xvi.
116 JACOBI 1842. These lectures were given by Jacobi at the University of Königsberg during the winter term 1842/1843. The material we are interested in here, is contained in the lectures 2-5. As to Jacobi see: LEJEUNE DIRICHLET 1881, and KÖNIGSBERGER 1904.
117 D'ALEMBERT 1743, paragraph 50. I had only the German translation available. As to D'Alembert see: BERTRAND 1889 and GRIMSLEY 1963.
118 Lagrange considers the more general case with external forces, too.

Considering displacements $\delta \mathbf{x}_i = \mathbf{a}$, Lagrange observes that $\delta r_{ik} = 0$, so that $\Sigma\, m_i \ddot{\mathbf{x}}_i = \frac{\mathrm{d}}{\mathrm{d}t} \Sigma\, m_i \dot{\mathbf{x}}_i = 0$, i.e. the total momentum $\Sigma m_i\, \dot{\mathbf{x}}_i$ is conserved.

Similarly, by considering the infinitesimal rotations $\delta x_i = -y_i \delta\varphi$, $\delta y_i = x_i\, \delta\varphi$, $\delta z_i = 0$ around the z- and the other axes which again imply $\delta r_{ik} = 0$, Lagrange derived the conservation of angular momentum for a closed system

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{n} m_i\, (\mathbf{x}_i \times \dot{\mathbf{x}}_i) = 0 \,.$$

Putting $\delta \mathbf{x}_i = \mathrm{d}\mathbf{x}_i$ and by assuming the sum $\Sigma \mathbf{F}_i \mathrm{d}\mathbf{x}_i$ to be a total differential, Lagrange obtains from D'Alembert's principle the conservation of the total energy («forces vives»).

Obviously D'Alembert's principle here appears as powerful as the closely related action principle, a property, Lagrange realized[119] quite clearly: «Un des avantages de la formule dont il s'agit est d'offrir immédiatement les équations générales qui renferment les principes ou théorèmes connus sous les noms de *conservation des forces vives*, de *conservation du mouvement du centre de gravité*, de *conservation des moments de rotation* ou *Principe des aires*, et de *Principe de la moindre quantité d'action*.»

Jacobi derived[120] the classical conservation laws in the same way as Lagrange, being analytically somewhat more detailed and stating explicitly that the potential U should not depend on the time t if energy conservation is to hold.

Even closer to the spirit of Noether's first theorem is Hamilton's derivation of momentum and angular momentum conservation from invariance under translations and rotations by using his «principal function»[121]

$$S\,[t,\, \mathbf{x}_i;\, \mathbf{b}_i = \mathbf{x}_i(t=0)] \;=\; \int_{(\mathbf{b}_1,\ldots,\mathbf{b}_n)}^{(\mathbf{x}_1,\ldots,\mathbf{x}_n)} (\mathbf{p}_i \mathrm{d}\mathbf{x}_i - H \mathrm{d}t) \,,$$

$$\delta S = -H\, \delta t \;+\; \sum_{i=1}^{n} (\mathrm{grad}_{\mathbf{x}_i} S) \delta \mathbf{x}_i \;+\; \sum_{i=1}^{n} \;(\mathrm{grad}_{\mathbf{b}_i} S) \cdot \delta \mathbf{b}_i \,,$$

$$\mathrm{grad}_{\mathbf{x}_i} S = \mathbf{p}_i \,, \quad \mathrm{grad}_{\mathbf{b}_i} S = -\mathbf{p}_2(t=0) \,.$$

Hamilton argues[122] that a simultaneous translation or rotation of the initial and final configurations $(\mathbf{b}_1, \ldots, \mathbf{b}_n)$ and $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ should not change δS, so that

119 *Mécanique Analytique* I, p. 257. The italics are Lagrange's!

120 Klein gives all the credit for the derivation of the classical conservation laws by means of D'Alembert's principles to Jacobi (KLEIN 1927, 56-57). This certainly is not justified! Lagrange is the one who deserves the credit!

121 In paragraph 6 of HAMILTON 1834, he actually uses the «characteristic» function $V = S + Ht$, $H = E = \mathrm{const.}$ This does not make any difference, however.

122 «...it evidently follows from the conception of our characteristic function V, that this function depends on the initial and final positions of the attracting or repelling points of a system, not as referred to any foreign standard, but only as compared to one another; and therefore that this function will not vary, if wihout making any real change in either initial or final configuration, or in the relation of these to each other, we alter at once all the initial and all the final positions of the points of the system, by any common motion, whether of translation or of rotation». See HAMILTON 1834, p. 112.

$$\sum_i \mathrm{grad}_{\mathbf{x}_i} S \;=\; \sum_i \mathbf{p}_i(t) = -\sum_i \mathrm{grad}_{\mathbf{b}_i} S = \sum_i \mathbf{p}_i(t=0)$$

as a consequence of the invariance of δS under infinitesimal translations $\delta \mathbf{x}_i \to \delta\, (\mathbf{x}_i + \mathbf{a})$, $\delta \mathbf{b}_i \to \delta(\mathbf{b}_i + \mathbf{a})$.

In the same way Hamilton derives angular momentum conservation as a consequence of the invariance of δS under infinitesimal rotations!

Hamilton and Jacobi were very close to Noether's general result in mechanics in the following sense: Both used[123] the fact that a solution $S(t,q;a)$ of the Hamilton-Jacobi equation which depends on a parameter a has the property that $\partial S/\partial a$ becomes a constant of motion for those extremals $q^j(t)$ which are transversal to the wave fronts $S(t,q;a) = \mathrm{const.}$, that is to say for which $p_j(t) = \partial_j S(t,q;a)$. Because $\mathrm{d}S(t,q(t))/\mathrm{d}t = L$, where L is the Lagrangian function, we have on the other hand

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \frac{\partial S}{\partial a} = \frac{\partial L}{\partial a}$$

Suppose now that a is the parameter of an infinitesimal transformation $\delta t = T(t,q)\delta a$, $\delta q^j = Q^j(t,q)\; \delta a$, then we have

$$\left.\frac{\partial S}{\partial a}\right|_{a=0} \;=\; \partial_t\, ST + \partial_j S Q^j = -HT + p_j\, Q^j \,,$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(-HT + p_j Q^j) = \frac{\partial L}{\partial a} \,.$$

Thus, if L is invariant under such a transformtion, then the quantity $-HT + p_j Q^j$ is a constant of motion. This simple reformulation of Hamilton's and Jacobi's theorem that $\partial S/\partial a$ is a constant of motion on the extremals, is exactly Noether's theorem in the case of mechanics![124] Jacobi also made systematic use of Poisson's brackets in order to calculate new constants of motion from two given ones: He calculated[125] the third angular momentum component as the Poisson bracket of the two first ones and he was the first to write down[126] the Lie algebra of the Euclidean group in 3 dimensions by means of Poisson brackets!

123 Hamilton considers only the initial positions $(\mathbf{b}_1, \ldots, \mathbf{b}_n)$ as parameters whereas Jacobi discusses the general case of arbrrary parameters on which the function S may depend. Jacobi's work on Hamilton's theory is contained in the following publications: JACOBI 1837a, 1838 and 1842. On the relationship between Hamilton's and Jacobi's approaches see: «Editorial Note II» by A.W. Conway and A.J. McConnell in vol. II (613-621) of HAMILTON'S *Papers*. PRANGE 1904, chs. B-D. Of considerable interest in this context is also the note VI («Sur les équations différentielles des problèmes de Mécanique, et la forme que l'on peut donner à leur intégrales») by J. Bertrand at the end of his edition of LAGRANGE 1853, t. I., 468-484. It is closely related to our discussion above.

124 This derivation of Noether's first theorem in the case of mechanics is discussed in: KASTRUP 1983.

125 JACOBI 1842, lecture 34.

126 JACOBI 1838, p. 114.

Another important notion in the context of our discussion is that of a «cyclic» or «ignorable» coordinate, the importance of which was first stressed by Edward John Routh[127] (1831-1907) and a little later[128] by Helmholtz:

If the Lagrangian function $L(t,q,\dot{q})$ does not depend on one of the coordinates q^j, say q^1, then if follows from the equations of motion that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^1} = 0 ,$$

that is to say the canonical momentum $p_1 = \partial L/\partial \dot{q}^1$ is a constant of motion. In other words: if the Lagrangian L is invariant under the translation $q^1 \to q^1 + a$ of the generalized coordinate q^1, then we have the constant of motion $p_1 = p_1(q,\dot{q})$.

If there are 2 cyclic coordinates, say q^1 and q^2, then we have the constants of motion p_1 and p_2 which are «in involution», that is to say their Poisson bracket vanishes. The ultimate goal as to the integration of a system with $2n$ degrees of freedom in phase space is to find such a coordinate system in which all n coordinates q^j are cyclic; for then all momenta p_j are constants and since the Hamilton function H now depends only on these constants the integration of the equations of motion is trivial![129]

Intimately related to the method of obtaining constants of motions by means of cyclic coordinates are a number of papers which appeared during the last decade of the 19th century in the wake of Lie's work[130] on the groups of motion and the conformal groups associated with the geodesics of a Riemannian manifold:

Among those involved were Paul Appell[131] (1855-1930), Gaston Darboux[132] (1842-1917), Paul Painlevé[133] (1863-1933) and René Liouville[134] (1856-1930) in France; Otto Staude[135] (1857-1930), Paul Stäckel[136] (1862-1919) and Adolf Kneser[137] (1862-1930) in Germany, and Tullio Levi-Civita[138] (1873-1941) and Guido Fubini[139] (1877-1954) in Italy.

127 ROUTH 1877, ch. IV, paragraph 20. As to Routh see: FORSYTH 1907..
128 HELMHOLTZ 1884.
129 Modern expositions of these ideas are: MOSER 1973, WHITEMAN 1977, and ARNOLD 1983.
130 LIE 1882.
131 APPELL 1890, 1891, 1892a and 1892b. As to Appell and his work see: Paul Appell, «Notice sur les travaux scientifiques», *Acta Mathematica 45* (1925) 161-285; BUHL 1931.
132 DARBOUX 1889. As to Darboux see: LEBON 1913, HILBERT 1917b and VOSS 1918.
133 PAINLEVÉ 1892a, 1892b, 1892c, 1892d, 1893, 1894a, 1894b, 1895, and 1896, 16è leçon. All these papers are reprinted in: PAINLEVÉ *Papers*, t. III, 290-328, 423-510, 513-611. On pp. 277-281 of this volume there is an evaluation of Painlevé's work on mechanics by A. Lichnerowicz. A list of publications on Painlevé's life and work as a mathematician and as a politician —he was minister during the first world war— is contained in: PAINLEVÉ *Papers*, vol. I, pp. 23-24. See also J. Hadamard, same volume, 37-73.
134 LIOUVILLE 1891, 1892a, 1892b, 1892c and 1895. As to Liouville see: LÉVY 1931. This Liouville is not to be confused with the famous Joseph Liouville (1809-1882).
135 STAUDE 1892, 1893a, and 1893b. As to Staude see: SCHUR 1931.
136 STÄCKEL 1891, 1893a, 1893b, 1894a, 1894b, 1897, and 1898. As to Stäckel see: PERRON 1920 and LOREY 1921.
137 KNESER 1894 and 1917. As to Kneser see: KOSCHMIEDER 1930.
138 LEVI-CIVITA 1896. As to Levi-Civita see: CARTAN 1942 and AMALDI 1946.
139 FUBINI 1903a, 1903b, 1904, and 1908. All these papers are reprinted under the numbers 9, 11, 16 and 40 in: FUBINI *Papers*. As to Fubini see: SEGRE 1954.

I cannot cover all the problems discussed by these authors in those papers quoted, but will only outline that part which is of interest in our context: An essential element of the work concerned is Jacobi's version[140] of the principle of least action for conservative systems: If $L = T - U$ does not depend on the time t explicitly, then $T + U = E = \text{const.}$. Suppose further that the kinetic energy has the form $T = \frac{1}{2} g_{ij}\, \dot{q}^i \dot{q}^j$ and that the coefficients $g_{ij}(q)$ and the potential $U(q)$ depend only on the coordinates q^j, not on the velocities $\dot{q}^j$.

For a given total energy E the Lagrangian L becomes $L = 2T - E$ and the action integral $\int (T-U)\mathrm{d}t$ may be replaced by $\int 2T\mathrm{d}t$. Eliminating the time t by observing that $T = E - U$ one obtains

$$2T = 2(\tfrac{1}{2}\, g_{ij}\, \dot{q}^i \dot{q}^j)^{1/2}\, (E-U)^{1/2}$$

and the Jacobian action integral

$$A_J = \int [2(E-U)]^{1/2}\, (g_{ij}\, \mathrm{d}q^i \mathrm{d}q^j)^{1/2}$$

is the same as that for the geodesics of an n-dimensional Riemannian manifold with the metric

$$\mathrm{d}s^2 = 2(E-U)\, g_{ij}\, \mathrm{d}q^i\, \mathrm{d}q^j$$

Thus the dynamical problem has become a part of Riemannian geometry! Notice that the curve parameter $\tau(t)$ of the geodesics is arbitrary.[141] We, therefore, may take it to be $\tau = q^1$. Then there are $n-1$ second order differential equations for the geodesics $q^i(q^1)$, $i = 2,\ldots,n$. In addition to the energy constant E the orbits will depend on $2n-2$ arbitrary constants of integration. Suppose now that

$$\delta q^j = \xi^j(q)\; \delta a$$

is the infinitesimal transformation of a 1-parameter transformation group which, for a fixed but otherwise arbitrary constant E, leaves the differential equations for the geodesics invariant (i.e. if q is an orbit, then $q + \delta q$ is an orbit, too).

Then one may ask the question: What does this invariance imply for the transformation properties of the potential $U(q)$ and the «kinetic» line element $\mathrm{d}\sigma^2 = g_{ij}(q)\; \mathrm{d}q^i \mathrm{d}q^j$?

The answer given by Painlevé, Staude, Stäckel and Kneser is the following:[142]

140 JACOBI, 1837b and 1842, lecture 6. A beautiful exposition of the relationship between Riemannian geometry and Jacobi's version of the action principle is given by DARBOUX 1887, vol. 2 (1889), livre V,hhs. VI-VIII.
141 Some subtleties concerning Jacobi's action are discussed by WINTNER 1941, paragraphs 171-184.
142 See footnotes 133, 135, 136 and 137. Staude discussed the cases $n = 2,3$ whereas Painlevé, Staeckel and Kneser considered more general cases.

The potential U has to be an invariant, i.e. $\xi^j \partial_j U = 0$, and the line element $d\sigma^2$ has to be transformed conformally!

If one chooses coordinates q^i, such that the action of the group is just a translation[143] of, say q^2, then we have $\partial_2 U = 0$, i.e. U is independent of q^2, and $d\sigma^2$ has the form

$$d\sigma^2 = e^{bq^2}\, \bar{g}_{ij}\, dq^i\, dq^j\,, \quad b = \text{const.}\,,$$

where the coefficients $\bar{g}_{ij}$ do not depend on q^2!

If $b=0$ then the action of the group is an isometry and kinetic and potential energies are independent of q^2 which, therefore, is a cyclic coordinate and we have a conservation law. This is again a special example for E. Noether's first theorem.

The authors mentioned above did not draw any conclusions concerning conservation laws —the conclusions in the last paragraph are mine. They were interested in the following problem: Suppose, there are two conservative mechanical systems, defined by the three quantities $d\sigma^2 = g_{ij}dq^i dq^j$, $U(q)$, E and $d\hat{\sigma}^2 = \hat{g}_{ij}d\hat{q}^i d\hat{q}^j$, $\hat{U}(\hat{q})$, $\hat{E}$ respectively. When is the totality of orbits of the first system in one-to-one correspondence to the totality of orbits of the second system?

The answer is that the following Darboux transformation[144] has to hold between the three corresponding quantities:

$$d\hat{\sigma}^2 = (cU+d)d\sigma^2, \quad \hat{U} = \frac{aU+b}{cU+d}, \quad \hat{E} = \frac{aE+b}{cE+d}$$

$$a, b, c, d \text{ constants with } ad-cb \neq 0.$$

In our context we are dealing with the special case $a=d=1$, $b=c=0$.

In 1890 Poincaré pointed out[145] the connection between the invariance of the equations of motion under time translations, space translations, rotations and special Galileo transformations, and energy conservation, momentum conservation, angular momentum conservation and uniforme center of mass motions, respectively, without actually deriving these conservation laws; but he was probably the first one who drew attention to the relationship between special Galileo invariance and the uniforme center of mass motion.

In 1897 Ignaz Schütz[146] (-1926) derived momentum conservation from the Galileo-covariance of energy conservation by using the fact —to speak in modern terms— that the commutator between the generators of time translations

143 That such a choice of coordinates is possible, was already proved by Lie: LIE and ENGEL 1888, vol. I. p. 49.
144 See footnote 132.
145 POINCARÉ 1890, ch. II paragraph 5, and 1892, t.I. paragraph 56.
146 SCHÜTZ 1897. This paper became well-known after Minkowski mentioned it in his famous lecture: MINKOWSKI 1908b. I found a small note saying that Schütz had been an assistant of Boltzmann and Voigt, that he died in Aug. 1926 and that he had been without a scientific position during the last 20 years of his life. No birthdate is mentioned: *Jahresber. d. deutschen Mathem. Vereinig. 37* (1928), appendix, p. 27.

and special Galileo transformations is the generator for translations. In 1900 Poincaré did the same,[147] without mentioning Schütz.

E. Noether quotes two papers[148] by Georg Hamel[149] (1877-1954). As these papers have been mentioned several times[150] by Eugene Wigner, it may be useful to indicate their content:

Hamel is concerned with the generalization of the following properties of the 3-dimensional rotator:

The equations of motion for the rotator take different forms, depending on the frame of reference[151]:

In the inertial, space fixed frame we have

$$\frac{d}{dt} J^I_i = M^I_i, \quad i=1, 2, 3,$$

where J^I_i, $i=1,2,3$, are the three components of the angular momentum and M^I_i those of the torque, all with respect to the inertial frame I.

If $\vec{\omega}$ is the angular velocity vector in the body fixed frame B, then in this frame the Euler equations of motion are

$$\frac{d}{dt} J^B_i + \sum_{j,k=1}^{3} \varepsilon_{ijk}\, \omega_j\, J^B_k = M^B_i, \quad J^B_i = \omega_i\, \Theta_i, \quad i=1,2,3,$$

where Θ_i, $i=1,2,3$ are the three principal moments of inertia.

Let ψ, φ, θ be the three Euler angles which describe the position of the body-fixed frame with respect to the speace-fixe frame, then the components ω_i are given by the following 3 «non-holonomic» relations

$$\begin{aligned}
\omega_1 &= \cos\varphi\ \dot{\theta} + \sin\theta\ \sin\varphi\ \dot{\psi} \\
\omega_2 &= -\sin\varphi\ \dot{\theta} + \sin\theta\ \cos\varphi\ \dot{\psi} \\
\omega_3 &= \dot{\varphi} + \cos\theta\ \dot{\psi}
\end{aligned}$$

Hamel notices that the structure of the dynamical equations is intimately related to the Lie algebra of the rotation group $SO(3)$:

i. The transformation from the variables ω_i, $i=1,2,3$, to the variables $\dot{\psi}$, $\dot{\varphi}$, $\dot{\theta}$, provides a representation of the Lie algebra of SO (3).

ii. The coefficients ε_{ijk} in the term $\varepsilon_{ijk}\ \omega_j J^B_k$ are the structure constants of the Lie algebra.

147 POINCARÉ 1900.
148 HAMEL 1904a, and 1904b.
149 As to Hamel see: KUCHARSKI 1952, SCHMEIDLER 1952 and 1955.
150 WIGNER 1949, footnote 4; 1954, ref. 1; 1964, ref. 4; 1963, footnote 14 (1967, footnote 17). HOUTAPPEL, VAN DAM and WIGNER 1965, footnote 20. This footnote —the main content of which is attributed to E. Guth— describes the story of the conservation laws in mechanics very much like Klein did: KLEIN 1927, 56-59. Klein's story is, however, very incomplete, see my footnote 120.
151 KLEIN and SOMMERFELD 1897.

iii. The simplicity of the dynamical equations is related to the rotational invariance of the kinetic energy of the rotator.

Hamel then generalizes the 3-dimensional rotator to an n-dimensional one, where the group $SO(n)$ replaces the $SO(3)$ and where the equations of motion in a body-fixed frame take the form

$$\frac{d}{dt} J_i^B + \sum_{j,k=1}^{n} C_{ijk}\, \omega_j J_k^B = M_i^B, \quad i=1,\dots,n .$$

Here C_{ijk} are the structure constants of the Lie algebra of the group $SO(n)$. Hamel mentions that the n-dimensional rotator may have cyclic coordinates, like the 3-dimensional one has. In addition he says explicitly[152] that his work is not related to that of Painlevé, Stäckel and Staude. He does not seem to know that the main results of his work are already contained in a brief but beautiful paper from 1901 by Poincaré,[153] in which the group $SO(3)$ is replaced by a transitive transformation group acting on an n-dimensional configuration space. Poincaré stresses the importance of cyclic coordinates, too.

We see that Emmy Noether had a number of predecessors as to the problem of relating symmetry properties to conservation laws for special systems, especially in mechanics. Hower, here again what van der Waerden said in his obituary[154] about her work is very true: «The maxim by which Emmy Noether was guided throughout her work might be formulated as follows: "Any relationships between numbers, functions and operations only become transparent, generally applicable, and fully productive after they have been isolated from their particular objects and been formulated as universally valid concepts."»

The beauty and the outstanding importance of Emmy Noether's paper from July 1918 consists in its combination of two properties: It is extremely general on the one hand, but on the other hand it provides an elementary construction of the conserved quantities, once the Lagrangian and its invariance group are given!

7. A few remarks on the recognition of Noether's theorems within the scientific community

> ... während die Physiker jetzt diese Begriffe zum Teil neu erfinden und sich durch einen Urwald von Unklarheiten mühevoll einen Pfad durchholzen müssen, indessen ganz in der Nähe die längst vortrefflich angelegte Straße der Mathematiker bequem vorwärts führt.
> Hermann Minkowski[155]

It took more than 30 years till Emmy Noether's work concerning the

152 HAMEL 1904a, footnote 4, on p. 4.
153 POINCARÉ 1901b. In a later paper Hamel acknowledges the priority of Poincaré and others: HAMEL 1908, p. 385.
154 See footnote 8.
155 MINKOWSKI 1907. English translation: «... while the physicists are now inventing these concepts anew and have to hack their trail through a jungle of confusions, the very well-built road of the mathematicians, near at hand, leads comfortably forward».

relationship between invariance properties and conservation laws of a physical system was fully appreciated. I shall indicate a few instances of this historical development until approximately 1960: I mentioned already that E. Bessel-Hagen, prodded by Felix Klein, applied Noether's results to mechanics and electrodynamics[156] in 1921.

In his extended paper «Die Grundlagen der Physik» from 1924 Hilbert drew attention[157] to E. Noether's paper from July 1918 and its second theorem. Her paper was discussed in some detail in the first volume of the textbook *Methoden der Mathematischen Physik* by Courant and Hilbert in 1924, too.[158] Hermann Weyl mentions her paper in a bibliographical note,[159] contained in the 4th edition (1921) of his textbook *Space-Time-Matter*.

The review article by R. Weitzenböck on the recent developments in algebraic invariant theory and differential invariants in the *Encyklopädie der Mathematischen Wissenschaften* from 1922 contained a summary, written by herself,[160] of E. Noether's papers from January and July 1918. Around that time recognition of her work seems to have been confined to the Göttingen circle! Wolfgang Pauli (1900-1958), in his famous relativity article[161] from 1921 mentions only her paper on differential invariants from January 1918.

Then quantum mechanics came with its emphasis on the Hamiltonian framework in mechanics and with its new formulation of symmetries as being associated with unitary (or antiunitary) representations of groups in the Hilbert spaces of states.[162] All three classical books on group theory and quantum mechanies, namely those by Hermann Weyl,[163] Eugene Paul Wigner[164] (1902-) and Bartel Laendert van der Waerden[165] (1903-), do not deal with action integrals and their invariance properties!

Only with the rise of quantum field theory and elementary particle physics did the Lagrangian framework slowly come back into view: The variational identity (1) was rediscovered several times by physicists[166] dealing with field theories, without mentioning Noether's work.

156 See footnote 52.
157 HILBERT 1915, 1917a and 1924.
158 COURANT and HILBERT 1924, Bd. I, 216-219.
159 WEYL 1922, p. 322, Note 5 of ch. IV.
160 WEITZENBÖCK 1927, paragraph 28; reprinted in: NOETHER *Papers*, 405-408.
161 PAULI 1921, p. 598, footnote 84. It is hardly necessary to quote literature «as to Pauli». Let me, nevertheless, mention the following: FIERZ and WEISSKOPF 1960, and HERMANN, MEYENN and WEISSKOPF 1979.
162 Representation theory of groups (initiated by the mathematicians G. Frobenius and I. Schur) was introduced into the new quantum mechanics —with some help by J. von Neumann— by Eugene P. Wigner: WIGNER 1927a and 1927b. About a year later a paper by H. Weyl followed: WEYL 1928b. On symmetries in classical and quantum systems and their relationships see, e.g., HOUTAPPEL, VAN DAM and WIGNER 1965.
163 WEYL 1928b.
164 WIGNER 1931.
165 VAN DER WAERDEN 1932.
166 Here are a few examples: HEISENBERG and PAULI 1929 and 1930. ROSENFELD 1940. BELINFANTE 1940a and 1940b. PAULI 1941. WENTZEL 1943. SCHWINGER 1951. Note added in proof: After having read the preprint of the present article of mine, Arthur S. Wightman, Princeton, in Jan. 1985 wrote me a letter concerning the content of this section 7: although it is true that theoretical physicists did not quote E. Noether' paper in the fourtieth, a number of them were quite aware of it.

For the thirties I could make out just *one* paper which explicitly and systematically applied Noether's first theorem to a known field theory! That is the paper by Moisei A. Markov[167] (1908-) about the currents of a Dirac particle in an external electromagnetic field. The subject of his article was suggested — according to Markov's remarks at the end — by Yuri B. Rumer[168] (1901-) who had been in Göttingen for several years in the late twenties and early thirties, part of that time as an assistant of Born.[169] He must have met E. Noether during that time and gotten acquainted with her and Bessel-Hagen's work!

Prompted by Markov's paper Noether's theorems were discussed in 1949 by Iwanenko and Sokolov in their textbook[170] on classical field theory. A breakthrough came with Edward L. Hill's (1904-1974) exposition[171] of Noether's paper in 1951. In a paper from 1952 P. G. Bergmann and R. Thomson[172] refer to Noether's second theorem. Beginning in 1956 there are several beautiful papers[173] by A. Trautman dealing with the problems associated with Noether's two theorems. In the first volume of their textbook on *Mesons and Fields*,[174] Bethe, Schweber and de Hoffmann mention Hill's paper. The *Introduction to the Theory of Quantized Fields*[175] from 1957, by Bogoliubov and Shirkov, has a paragraph entitled «Noether's theorem», without quoting her paper. The first volume on *Field Theory*[175] from 1958 by Rzewuski has a long discussion on the connection between the invariance of the action integral and the conservation laws of a system, quoting Noether's paper. So does Roman in his *Theory of Elementary Particles*[177] from 1960.

Thus, the physics community slowly became aware of that important piece of work from 1918 by an important mathematician, who, however, was treated badly during her lifetime by the German society, first — with the exception of Hilbert and Weyl[178] — by the scientific establishment, because she was a woman, and then in 1933 by the German state, because she was a Jew.

167 MARKOW 1936. As to Markow see: BALDIN and KOMAR 1978.
168 As to Rumer see: POGGENDORFF *Biogr.* (1958), Bd. VIIa, Teil 3 (L-R), p.850. See also footnote 169. In 1931 Rumer himself had written a review on the status of the Dirac theory: RUMER 1931. In 1937/38 Rumer collaborated with Landau, see papers nr. 28, 34, and 36 in LANDAU *Papers* and RUMER 1973.
169 See: BORN 1969, letters 59-62, 64, 65.
170 IWANENKO and SOKOLOW, 1949, paragraph 23.
171 HILL 1951. As to Hill see: *American Men and Women of Science*, 12th ed. vol. 3 (H—K) (Jackes Cattel Press/R.R. Bowker and Co., New York and London 1972), p. 2706. *Physics Today 27* (1974), July, p. 59.
172 BERGMANN and THOMSON 1953.
173 TRAUTMAN 1956a, 1956b, 1957, 1962, and 1965, ch.7,
174 SCHWEBER, BETHE and DE HOFFMANN, 1955, section 10b.
175 BOGOLIUBOV and SHIRKOV 1959, paragraph 2.5.
176 RZEWUSKI 1958, ch. II, 2. See also: RZEWUSKI 1953.
177 ROMAN 1960.
178 See the literature in footnote 8.

8. Acknowledgements

> Doch ist in dieser schwer verständlichen, oft fast unauffindbaren, in der ganzen Welt zerstreuten mathematischen Literatur ungemein viel des Brauchbaren, auch für den Praktiker Nützlichen, ja fast Unentbehrlichen, vergraben.
> Ludwig Boltzmann[179] (1844-1906)

I am very much indebted to several old reviews for helping me to find the literature mentioned in the previous chapters: As to the publications on mathematical problems in mechanics in the first half of the 19th century, Cayley's two reports[180] from 1857 and 1862 are very valuable. Many historical references are contained in the review articles by Aurel Voss[181] (1845-1931) and Paul Stäckel on the principles of mechanics in vol. IV,1 of the *Encyklopädie der Mathematischen Wissenschaften*.[182] Then there are, of course, Felix Klein's lectures[51,52] on the history of mathematics in the 19th century. Of considerable help was the survey of the general methods of integration in mechanics by Georg Prange[183] (1885-1941) in vol. IV,2 of the *Encyklopädie der Mathem. Wissenschaften*.[184] Here I learnt about the work of Darboux, Painlevé, Staude, Stäckel, Kneser and others discussed in my chapter 6 above. Quite valuable are the numerous historical notes and references in the appendix of Wintner's book[112] on celestical mechanics.

As I had to mention many scientists, the life (or work) of whom is no longer well-known, I have added for each of them one or more references to articles — many of them obituaries —, from which one can learn more about these people and their work. For finding those articles the following review journals were very helpful: *Jahrbuch über die Fortschritte der Mathematik* (from 1868 to 1942) and *Zentralblatt für Mathematik und ihrer Grenzgebiete* (since 1931).

Finally I would like to mention a book which is somewhat outside the scope of my previous chapters, but which deals with many topics discussed there in such a

179 BOLTZMANN, «Reise eines deutschen Professors ins Eldorado», in: BOLTZMANN 1979, 258-290, here p. 260. This is a humorous, sometimes sarcastic account (from 1905) of Boltzmann's trip to and through the United States. On his way from Vienna to Bremen he passed through Leipzig in order to attend a meeting of the editors and collaborators of the huge project of the *Encyklopädie der Mathematischen Wissenschaften*, initiated, coordinated and pushed by Felix Klein as the editor in chief. Boltzmann makes witty observations and ironical comments on the whole enterprise and on the idiosyncrasies of the people involved. The above quotation is from one of the sections which describe the meeting in Leipzig. I am indebted to Friedrich Hehl, Cologne, for drawing my attention to this entertaining piece of Boltzmann's prose. Here is a translation: «However, an immense amount of usable material, also useful and in fact even almost indispensable to the practitioner, is buried in this mathematical literature, which is difficult to understand, often almost untraceable and scattered all over the world». Boltzmann himself, together with J. Nabl contributed the following article to the *Encyklopädie* «Kinetische Theorie der Materie»; in: *Enc. Math. Wiss.*, Bd. V.1 (B. G. Teubner, Leipzig 1903-1921) 493-557. As to Boltzmann see the contributions by D. Flamm, E. Broda and M.J. Klein in: COHEN and THIRRING 1973. See also: KUHN 1978.
180 CAYLEY 1857, 1862.
181 As to Voss see: POGGENDORFF *Biogr.*, Bd. VI, IV. Teil S-Z (1940) p. 2779.
182 VOSS 1901 and STÄCKEL 1901.
183 As to Prange see: KOPPENFELS 1941. Prange does not mention Noether's and Bessel-Hagen's papers in his long review from 1933.
184 See footnote 123.

superior, competent and original manner that it should not remain unmentioned: Élie Cartan's *Leçons sur les invariants intégraux*![185]

Last but not least I thank my wife, Barbara, for patiently correcting the worst of my attempts to translate German quotations into English!

Without the invitation of M.G. Doncel and A. Hermann to come to the conference in Sant Feliu de Guixols this paper would not have been written. So I thank both of them, too. I enjoyed the conference very much!

185 CARTAN 1922.

References

ALEXANDROFF, P.S.
1936 «In memory of Emmy Noether»; *Uspekhi Mat. Nauk 2* (1936) 254-266; English translations in: DICK 1981, 153-179, in: BREWER and SMITH 1981, 99-111, and in: NOETHER *Papers*, 1-11.

AMALDI, U.
1946 «Commemorazione del socio Tullio Levi-Civita»; *Atti della Accademia Nazionale dei Lincei, Rendiconti, Cl. di Scienze Fisiche, Matem. e Nat.* (Ser. 8) *1* (1946) 1130-1155.

APPELL, P.
1890 «De l'homographie en mécanique»; *Amer. Journ. of Mathematics 12* (1890) 103-114.
1891 «Sur les lois de forces centrales faisant décrire à leur point d'application une conique quelles que soient les conditions initiales»; *Amer. Journ. of Math. 13* (1891) 153-158.
1892a «Sur des transformations de mouvements»; *Journ. f. d. reine u. angew. Math. 110* (1892) 37-41.
1892b «Sur une transformation de mouvement et les invariants d'un système en Mécanique»; *Bull. de la Soc. Mathématique de France 20* (1892) 21-22.

ARNOLD, V. I.
1983 *Geometrical Methods in the Theory of Ordinary Differential Equations* (Grundlehren der math. Wissensch. *250*; Springer-Verlag; New York, Heidelberg, Berlin, 1983).

BAKER, H.F.
1912 «Biographical Notice»; in: SYLVESTER *Papers*, vol. IV (1912) xiii-xxxvii

BALDIN, A.M. and A.A. KOMAR
1978 «Moisei Aleksandrovich Markov (on his seventieth birthday)»; *Usp. Fiz. Nauk 125* (1978) 363-368; Engl. translation: *Sov. Phys. Usp. 21* (6) (1978) 544-548.

BEHNKE, HEINRICH
1960 «Felix Klein und die heutige Mathematik»; *Mathem.—Physikal. Semesterberichte 7* (1960/1961) 129-144.

BEHNKE, H. and K. KOPFERMANN (eds.)
1966 *Festschrift zur Gedächtnisfeier für Karl Weierstrass 1815-1965* (Westdeutscher Verlag, Köln und Opladen 1966).

BELINFANTE, F.J.
1940a «On the current and the density of the electric charge, the energy, the linear momentum and the angular momentum of arbitrary fields»; *Physica 7* (1940) 449-474.
1940b «On the quantum theory of wave fields»; *Physica 7* (1940) 765-778.

BELTRAMI, E.
1868 «Sulla teorica generale dei parametri differenziali»; *Memorie dell'Academia delle Scienze dell'Istituto di Bologne* (ser. II) *8* (1868) 551-590; reprinted in: *Opere Matematiche di Eugenio Beltrami*, t. II (U. Hoepli, Milano 1904) 74-118.

BERGMANN, PETER G. and ROBB THOMSON
1953 «Spin and Angular Momentum in General Relativity»; *Phys. Review 89* (1953) 400-407.

BERTRAND, J.
1889 *D'Alembert* (Hachette, Paris 1889).

BESSEL-HAGEN, ERICH
1921 «Über die Erhaltungssätze der Elektrodynamik»; *Mathem. Ann. 84* (1921) 258-276.

BIERMANN, KURT R.
1966 «Karl Weierstrass, Ausgewählte Aspekte seiner Biographie»; *Journ. f. d. reine u. angew. Math. 223* (1966) 191-220.

BLUMAN, G. W. and J.D. COLE
1974 *Similarity Methods for Differential Equations* (Applied Mathematical Sciences 13; Springer-Verlag; New York, Heidelberg, Berlin, 1974).

BOGOLIUBOV, N.N. and D.V. SHIRKOV
1959 *Introduction to the Theory of Quantized Fields* (Interscience Publ., New York and London 1959).

BOLTZMANN, LUDWIG
1979 *Populäre Schriften*, eingeleitet und ausgewählt von Engelbert Broda (Fr. Vieweg and Sohn, Braunschweig/Wiesbaden 1979)

1982 *Vorlesungen über Maxwells Theorie der Electricität und des Lichtes, I. und II. Teil;* Einleitung, Anmerkungen und Bibliographie von Walter Kaiser (*Ludwig Boltzmann Gesamtausgabe,* hrsg. von Roman U. Sexl, Bd. 2; Akademische Druck-u. Verlagsanstalt, Graz, und Friedr. Vieweg u. Sohn, Braunschweig/Wiesbaden 1982).

BORN, MAX
1909 «Die träge Masse und das Relativitätsprinzip»; *Annalen der Physik* (4. Folge) *28* (1909) 571-584.
1928 «Antoon Lorentz»; *Nachr. v. d. Ges. d. Wiss. zu Göttingen* 1928, 69-73; reprinted in: BORN *Papers,* vol. II, 607-611
1959 «Erinnerungen an Hermann Minkowski zur 50. Wiederkehr seines Todestages»; *Die Naturwissenschaften 46* (1959) 501-505; reprinted in: BORN *Papers,* vol. II, 678-690.
1968 My Life and my Views (Ch. Scribner's Sons Ltd., London 1968).
1969 *Hedwig und Max Born, Briefwechsel 1916-1955, kommentiert von Max Born* (Nymphenburger Verlagshandlung, München 1969); English translation publ. by Walker, New York 1971.
Papers Ausgewählte Abhandlungen, 2 vols. (Vandenhoek und Ruprecht, Göttigen 1963).

BREWER, JAMES W. and MARTHA K. SMITH (eds.)
1981 *Emmy Noether, a Tribute to her Life and Work* (Marcel Dekker, Inc., New York and Basel 1981).

BRILL, A.
1923 «Max Noether»; *Jahresbericht der deutschen Mathematiker-Vereinigung 32* (1923) 211-233

BRILL, A., P. GORDAN, F. KLEIN, J. LÜROTH, A. MAYER, E. NOETHER and K. VON DER MÜHLL
1874 «Rudolf Friedrich Alfred Clebsch»; *Mathem. Annalen 7* (1874) 1-55.

BROWDER, F. (ed.)
1983 *The Mathematical Heritage of Henri Poincaré,* 2 vols. (Americ. Math. Soc., Providence, L. I., 1983)

BUHL, A.
1931 «Paul Appell»; *L'Enseignement Mathématique 30* (1931) 5-21.

BRUNET, PIERRE
1938 *Etude historique sur le Principe de la Moindre Action* (Actualités Scientifiques et Industrielles *693*; Hermann et C^{ie}. Paris 1938).

BURZIO, FILIPPO
1942 *Lagrange* (Unione tip. ed. torinese, Torino 1942).

BUTZER, P. L. and F. FÉHER (eds.)
1981 *E. B. Christoffel, The Influence of his Work on Mathematics and the Physical Sciences* (Birkhäuser Verlag, Basel, Boston, Stuttgart, 1981).

CARTAN, ÉLIE
1922 *Leçons sur les invariants intégraux* (Hermann, Paris, 1922 and 1971).
1942 «Notice sur Tullio Levi-Civita»; *C. R. Acad. Sc. 215* (1942) 233-235.

CASTELNUOVO, G., F. ENRIQUES and F. SEVERI
1925 «Max Noether»; *Mathem. Annalen 93* (1925) 161-181.

CAYLEY, A.
1845 «On the theory of linear transformations»; *Cambridge Mathematical Journal 4* (1845) 193-209; reprinted in: CAYLEY *Papers,* vol. I, 80-94.
1846a «On linear transformations»; *Cambridge and Dublin Mathematical Journal 1* (1846) 104-122; reprinted in: CAYLEY *Papers,* vol. I, 95-112.
1846b «Mémoire sur les Hyperdéterminants»; *Journal für die reine und angewandte Mathematik 30* (1846) 1-37.
1857 «Report on the recent progress of theoretical dynamics»; *Report of the British Association for the Advancement of Science 1857,* 1-42; reprinted in: CAYLEY *Papers,* vol. III, 156-204.
1862 «Report on the progress of the solution of certain special problems in dynamics»; *ReporT of the British Association for the Advancement of Science 1862,* 184-252; reprinted in: CAYLEY Papers, vol. IV, 513-593.
Papers The Collected Mathematical Papers, 13 vols. (Cambridge Univ. Press, Cambridge 1889-1898).

CHEVALLEY, C. and A. WEIL
1957 «Hermann Weyl (1885-1955)»; *L'Enseignement Mathématique* (II^e Sér.) *3* (1957) 157-187; reprinted in WEYL *Papers,* Bd. IV (1968) 655-785.

CHRISTOFFEL, E.B.
1869 «Über die Transformation der homogenen Differentialausdrücke zweiten Grades»; *Journal für die reine und angewandte Mathematik 70* (1869) 46-70, 241-245.

CLEBSCH, ALFRED
1895 «Zum Gedächtnis an Julius Plücker»; in: *Julius Plückers gesammelte wissensch. Abhandlungen Bd. I* (B. G. Teubner, Leipzig 1895) IX-XXXV.

COHEN, ABRAHAM
1911 *An Introduction to the Lie Theory of One-Parameter Groups* (D.C. Heath, Boston 1911).

COHEN, E.G.D. and W. THIRRING (eds.)
1973 *The Boltzmann Equation, Theory and Applications* (Springer-Verlag, Wien, New York, 1973).

COURANT, R.
1925 «Felix Klein»; *Die Naturwissenschaften 13* (1925) 765-772 and *Jahresber. d. deutschen Mathematiker-Vereinigung 34* (1926) 197-213.
1926 «Bernhard Riemann und die Mathematik der letzten hundert Jahre»; *Die Naturwissenschaften 14* (1926) 813-818, 1265-1277.

COURANT, R. and D. HILBERT
1924 *Methoden der Mathematischen Physik* (J. Springer, Berlin 1924).

D'ALEMBERT, J. LEROND
1743 *Traité de Dynamique* (Paris 1743); German translation: «D'Alembert, Abhandlung über Dynamik», übersetzt und herausgegeben von A. Korn; *Ostwald's Klassiker der exakten Naturw. 106* (W. Engelmann, Leipzig 1899).

DARBOUX, GASTON
1887 *Leçons sur la Théorie Générale des Surfaces,* 4 vols. (Gauthier-Villars, Paris 1887-1896).
1889 «Remarque sur la Communication précedente (Note de E. Goursat)»; *C. R. Acad. Sc. 108* (1889) 449-450.

DE HAAS-LORENTZ, G.L. (ed.)
1957 *H. A. Lorentz, Impressions of his Life and Work* (North-Holland Publ. Co., Amsterdam 1957).

DICK, AUGUSTE
1981 *Emmy Noether 1882-1935* (Birkhäuser; Boston, Basel, Stuttgart, 1981).

DIEUDONNÉ, JEAN A.
1972 «The historical development of algebraic geometry»; *The American Mathematical Monthly 79* (1972) 827-866

DIEUDONNÉ, JEAN A. and JAMES B. CARRELL
1970 «Invariant Theory, Old and New»; *Advances in Mathematics 4* (1970) 1-80.

EDDINGTON, A. S.
1942 «Sir Joseph Larmor»; *Nature 149* (1942) 631-633.

EGUCHI, T., P.B. GILKEY and A.J. HANSON
1980 «Gravitation, gauge theories and differential geometry»; *Physics Reports 66* (1980) 213-393

EINSTEIN, ALBERT
1905 «Zur Elektrodynamik bewegter Körper»; *Annalen der Physik* (4. Folge) *17* (1905) 891-921.
1915a «Zur allgemeinen Relativitätstheorie» (mit Nachtrag); *Sitzungsberichte der königlich preussischen Akademie der Wissenschaften zu Berlin* 1915, 778-786, 799-801.
1915b «Die Feldgleichungen der Gravitation»; *Sitzungsberichte der königl. preuss. Akad. d. Wiss. zu Berlin* 1915, 844-847.
1973 *Ideas and Opinions* (Souvenir Press [Educational and Academic] Ltd., London 1973).

EISENSTEIN, GOTTHOLD
Papers Gotthold Eisenstein, Mathematische Werke, 2 vols. (Chelsea Publishing, New York 1975).

ENGEL, FRIEDRICH
1899a «Sophus Lie»; *Berichte über die Verhandl. d. königl. sächs. Ges. d. Wiss. zu Leipzig, Math.-Phys. Kl. 51* (1899) XI-LXI.
1899b «Sophus Lie»; *Jahresber. d. deutschen Mathem.-Vereining. 8* (1899) 30-46
1916 «Über die zehn allgemeinen Integrale der klassischen Mechanik»; *Nachr. v. d. Königl. Gesellsch. d. Wiss. zu Göttingen, Math.-phys. Kl.* 1916, 270-275.
1917 «Nochmals die allgemeinen Integrale der klassischen Mechanik»; *Nachr. v.d. königl. Ges. d. Wiss. zu Göttingen* 1917, 189-198.

1938 «Friedrich Engel»; in: *Deutsche Mathematik 3* (1938) 701-719 (list of publications with comments by Engel himself).

ENGEL, FRIEDRICH and KARL FABER

1932 *Die Liesche Theorie der partiellen Differentialgleichungen erster Ordnung* (Verlag B.G. Teubner, Leipzig und Berlin 1932).

FABER, KARL and EGON ULLRICH

1945 «Gedenkband für Friedrich Engel»; *Mitteilungen des Mathematischen Seminars der Universität Giessen*, Heft 34-36 (1945).

FIERZ, M. and V.F. WEISSKOPF (eds.)

1960 *Theoretical Physics in the Twentieth Cnetury.* A Memorial Volume to Wolfgang Pauli (Interscience Publ., New York 1960).

FISCHER, CHARLES S.

1966 «The Death of a Mathematical Theory: a Study in the Sociology of Knowledge»; *Archive for History of Exact Sciences 3* (1966/1967) 137-159.

FORSYTH, A.R.

1895 «Arthur Cayley», in: CAYLEY *Papers*, vol. VIII (1895) ix-xliv.

1907 «Edward John Routh»; *Proceed. of the London Math. Soc.* (2nd series) (1907) xiv-xx.

FROBENIUS, FERDINAND GEORG

1893 «Gedächtnisrede auf Leopold Kronecker»; *Sitzungsber. d. königl. preuss. Akad. d. Wiss. zu Berlin* 1893, 3-22; reprinted in: FROBENIUS *Papers*, vol. III, 705-724.

Papers Gesamm. Abhandl., 3 vols., *herausgegeben von J.-P. Serre* (Springer-Verlag; Berlin, Heidelberg, New York, 1968).

FUBINI, GUIDO

1903a «Sui gruppi di trasformazioni geodetiche»; *Memorie dell'Acc. delle Scienze di Torino* (ser. 2) *53* (1903) 261-313.

1903b «Ricerche gruppali relative alle equazioni della dinamica»; *Atti della Reale Accademia dei Lincei, Rendic. d. Cl. Sc. Fis. Mat. Nat.* (ser. 5) *12* (1903), 1.º sem., 502-506; 2.º sem., 60-67, 145-151.

1904 «Sulle Traiettorie di un problema dinamico»; *Rend. del Circ. mat. di Palermo 18* (1904) 301-310.

1908 «Applicazioni della teoria dei gruppi continui alla geometria differenziale e alle equazioni di Lagrange»; *Math. Ann. 66* (1908) 202-214.

Papers Opere Scelte, vols. I and II (Edizioni Cremonese, Roma 1957, 1958).

GORDAN, PAUL

1885 *Vorlesungen über Invariantentheorie, 2 Teile, herausgegeben von G. Kerschensteiner* (Teubner, Leipzig 1885 and 1887).

GOURSAT, EDOUARD

1921 *Leçons sur l'Intégration des Équations aux Dérivées Partielles du Premier Ordre*, 2ᵉ éd. (J. Hermann, Paris 1921).

GRAVES, R. P.

1882 *Life of Sir W. R. Hamilton*, 3 vols. (Dublin Univ. Press, Dublin 1882-1889).

GRIMSLEY, R.

1963 *Jean d'Alembert (1717-1783)* (Clarendon PressOOxford 1963).

HALPHEN, G.-H.

Papers Oeuvres, publiées par les soins de C. Jordan, H. Poincaré, E. Picard, 4 vols. (Gauthier-Villars, Paris 1916-1924).

HAMEL, GEORG

1904a «Die Lagrange-Eulerschen Gleichungen der Mechanik»; *Zeitschr. f. Mathematik u. Physik 50* (1904) 1-57.

1904b «Über die virtuellen Verschiebungen in der Mechanik»; *Math. Ann. 59* (1904) 416-434.

1908 «Über die Grundlagen der Mechanik». *Mathem. Ann 66* (1980) 350-397.

HAMILTON, W.R.

1834 «On a general method in dynamics»; *Philosoph. Transactions of the Royal Soc. of London 1834* (part II) 247-308, paragraph 6; reprinted in: HAMILTON *Papers*, vol. II, 103-161.

Papers The Mathematical Papers of Sir William Rowan Hamilton, 3 vols. (Cambridge Univ. Press, Cambridge 1931, 1940 and 1967).

HARTMAN, P.

1962 «Aurel Wintner»; *The Journ. of the London Mathem. Soc. 37* (1962) 483-503.

HEISENBERG, WERNER

1970 «Max Born zum Gedächtnis»; *Physikalische Blätter 26* (1970) 49-54.

HEISENBERG, W. and W. PAULI

1929 «Zur Quantendynamik der Wellenfelder»; *Zeitschr. f. Phys. 56* (1929) 1-61.

1930 «Zur Quantentheorie der Wellenfelder II»; *Zeitschr. f. Phys. 59* (1930) 168-190.

HELMHOLTZ, HERMANN VON

1884 «Studien zur Statik monocyklischer Systeme»; *Sitzungsber. d. königl. preuss. Akad. d. Wiss. zu Berlin 1884*, 159-177; reprinted in: HELMHOLTZ *Papers*, vol. III, 119-141.

1887 «Zur Geschichte des Princips der kleinsten Action»; *Sitzungsberichte d. königl. preuss. Akad. d. Wiss. zu Berlin 1887*, 225-236; reprinted in: HELMHOLTZ *Paper*, vol. III, 249-263.

1892 «Das Princip der kleinsten Wirkung in der Elektrodynamik»; *Annalen der Physik und Chemie (Neue Folge) 47* (1892), 1-26; reprinted in: HELMHOLTZ *Papers*, vol. III, 476-504, 597-603.

Papers Wissenschaftliche Abhandlungen, 3 vols. (J. A. Barth, Leipzig 1882-1895).

HERGLOTZ, GUSTAV

1911 «Über die Mechanik des deformierbaren Körpers vom Standpunkt der Relativitätstheorie»; *Annalen der Physik* (4. Folge) *36* (1911) 493-533; reprinted in: HERGLOTZ *Papers*, 258-298.

Papers Gesammelte Schriften, hg. von H. Schwerdtfeger (Vandenhoek und Ruprecht, Göttingen 1979).

HERMANN, A., K. VON MEYENN and V. F. WEISSKOPF (eds.)

1979 *Wolfgang Pauli, Scientific Correspondence with Bohr, Einstein, Heisenberg, a. O.*, vol. I: 1919-1929 and vol. II: 1930-1939 (Springer-Verlag, New York, Heidelberg, Berlin 1979 and 1985).

HILBERT, DAVID

1910 «Hermann Minkowski»; *Math. Ann. 68* (1910) 445-471; reprinted in: MINKOWSKI *Papers*, vol. I, V-XXXI, and in: HILBERT *Papers*, vol. III, 339-364.

1915 «Die Grundlagen der Physik (Erste Mitteilung)»; *Nachr. von der Königl. Gesellsch. d. Wiss. zu Göttingen, Mathem.-phys. Kl.* 1915, 395-407.

1917a «Die Grundlagen der Physik (Zweite Mitteilung)»; *Nachr. von der Königl. Gesellschaft d. Wiss. zu Göttingen, Math.-phys. Kl.* 1917, 53-76.

1917b «Gaston Darboux»; *Nachr. v. d. Königl. Ges. d. Wiss. zu Göttingen* 1917, 71-75; reprinted in: HILBERT *Papers*, vol. III, 365-369; French translation: *Acta Mathematica 42* (1920) 269-273.

1924 «Die Grundlagen der Physik»; *Mathem. Ann. 92* (1924) 1-32; reprinted in: HILBERT *Papers*, vol. III, 258-289.

Papers Gesammelte Abhandlungen, 3 vols. (Springer-Verlag, Berlin, Heidelberg, New York, 1932-1935 and 1970).

HILL, E.L.

1951 «Hamilton's Principle and the Conservation Theorems of Mathematical Physics»; *Rev. Mod. Physics 23* (1951) 253-260.

HÖLDER, O.

1896 «Über die Principien von Hamilton und Maupertuis»; *Nachr. v. d. Königl. Ges. d. Wiss. zu Göttingen, Math.-phys. Kl.* 1896, 122-157.

HOUTAPPEL, R. M. F., H. VAN DAM and E. P. WIGNER

1965 «The Conceptual Basis and Use of the Geometric Invariance Principles»; *Reviews Modern Phys. 37* (1965) 595-632.

IWANENKO, D. and A. SOKOLOW

1949 *Klassische Feldtheorie* (Akademie-Verlag, Berlin 1953; German translation of the Russian edition from 1949).

JACOBI, C. G. J.

1837a «Über die Reduction der Integration der partiellen Differentialgleichungen erster Ordnung zwischen irgend einer Zahl Variablen auf die Integration eines einzigen Systems gewöhnlicher Differentialgleichungen»; *Journ. f. d. reine u. angew. Math. 17* (1837) 97-162; reprinted in: JACOBI *Papers*, vol. IV, 57-127. French translation: *Journal de Mathématiques Pures et Appliquées 3* (1838) 60-96, 161-201.

1837b «Note sur l'intégration des équations différentielles de la dynamique»; *C. R. Acad. Sc. 5* (1837) 61-67; reprinted in: JACOBI *Papers,* vol. IV, 129-136.

1838 «Nova methodus, aequationes differentiales partiales primi ordinis inter numerum variabilium quemcunque propositas integrandi»; manuscript from 1838, published posthumously by Clebsch in *Journ. f. d. reine u. angew. Math. 60* (1862) 1-181; reprinted in: JACOBI *Papers,* vol. V, 1-189; German translation: «Neue Methode zur Integration partieller Differentialgleichungen erster Ordnung zwischen irgend einer Anzahl von Veränderlichen», herausgeg. von G. Kowalewski, *Ostwalds Klassiker der exakten Naturwiss. 156* (W. Engelmann, Leipzig 1906).

1842 «Vorlesungen über Dynamik» (winter term 1842/1843) in: JACOBI *Papers,* Supplementband.

Papers Gesammelte Werke, 8 vols. (G. Reimer, Berlin 1881-1891; reprinted by Chelsea Publ. Co., New York 1969).

JORDAN, CAMILLE

1867 «Sur les groupes de mouvements»; *Comptes rendus des séances de l'Académie des Sciences 65* (1867) 229-232; reprinted in: JORDAN *Papers,* vol. IV, 113-115.

1868 «Mémoire sur les groupes de mouvements»; *Annali di Matematica pura ed applicata 2* (1868/1869) 167-215, 322-345; reprinted in: JORDAN *Papers,* vol. IV, 231-302.

1870 *Traité des Substitutions et des Équations Algébriques* (Gauthier-Villars, Paris 1870).

Papers Oeuvres, 4 vols. (Gauthier-Villars, Paris 1961-1964).

JOURDAIN, PHILIP E.B.

1908a «On those Principles of Mechanics which depend upon Processes of Variation»; *Math. Ann. 65* (1908) 513-527.

1908b Editorial comments in: «Abhandlungen über die Prinzipien der Mechanik von Lagrange, Rodrigues, Jacobi und Gauss»; *Ostwalds Klassiker der exakten Naturwissenschaften 167* (W. Engelmann, Leipzig 1908) 31-68.

1913 *The Principle of Least Action* (Open Court Co., Chicago 1913).

1914 Editorial comments in: «Abhandlungen über jene Grundsätze der Mechanik, die Integrale der Differentialgleichungen liefern, von Isaac Newton (1687), Daniel Bernoulli (1745) und (1748) und Patrick d'Arcy (1747)»; *Ostwalds Klassiker der exakten Naturwissenschaften 191* (W. Engelmann, Leipzig und Berlin 1914) 76-109.

KAISER, WALTER.

1982 Editorial introduction and comments in: BOLTZMANN 1982.

KAST, W.

1957 «Gustav Mie»; *Physikalische Blätter 13* (1957) 129-131.

KASTRUP, H.A.

1983 «Canonical theories for Lagrangian dynamical systems in physics», *Physics Reports 101* (1983) 1-167.

KLEIN, FELIX

1872 *Vergleichende Betrachtungen über neuere geometrische Forschungen. Programm zum Eintritt in die philosophische Fakultät und den Senat der Friedrich-Alexanders-Universität zu Erlangen* (A. Deichert, Erlangen 1872); reprinted in *Math. Ann. 43* (1893) 63-100 and, with many notes and comments by Klein himself, in: KLEIN *Papers,* vol. I, 460-497.

1893 «Comparative review of recent researches in geometry»; *Bull. of the New York Math. Soc. 2* (1893) 215-249 (English Translation of «Vergleichende Betrachtungen....»).

1894 «Riemann und seine Bedeutung für die Entwicklung der modernen Mathematik»; *Jahresbericht der Deutschen Mathematiker-Vereinigung 4* (1894/1895) 71-87; reprinted in: KLEIN *Papers,* vol. III, 482-497; English translation: *Bull. of the Americ. Math. Soc. 1* (1895) 165-180.

1917 «Zu Hilberts erster Note über die Grundlagen der Physik»; *Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen, Mathem.-physik. Kl.* 1917, 469-482; reprinted —with additional comments by Klein himself— in: KLEIN, *Papers* vol. I, 553-567.

1918a «Über die Differentialgesetze für die Erhaltung von Impuls und Energie in der Einsteinschen Gravitationstheorie»; *Nachr. v. d. Königl. Gesellsch. d. Wiss. zu Göttingen, Math.-phys. Kl.* 1918, 235-253; reprinted in: KLEIN, *Papers* vol. I, 568-585.

1918b «Über die Integralform der Erhaltungssätze und die Theorie der räumlich-geschlossenen Welt»; *Nachr. v. d. Königl. Gesellsch. d. Wiss. z. Göttingen, Math.-phys. Kl.* 1918, 394-423; reprinted in: KLEIN *Papers,* vol. I, 586-612.

1923 «Göttinger Professoren. Lebensbilder von eigener Hand. Felix Klein»: in: *Mitteilungen des Universitätsbundes Göttingen, Jahrg. 5,* Heft 1 (1923) 11-36 (autobiographical notes by Klein).

1926 *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert,* Teil I (Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, Bd. 24; Verlag J. Springer, Berlin 1926 and 1979; reprinted by Chelsea Publ. Co. 1950.)

1927 *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert,* Teil II (Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, Bd. 25; Verlag J. Springer, Berlin 1927 and 1979; reprinted by Chelsea Publ. Co., New York 1950).

Papers Gesammelte mathematische Abhandlungen, 3 vols. (J. Springer, Berlin 1921-1923)

KLEIN, FELIX and SOPHUS LIE

1870a «Sur une certaine de courbes et de surfaces»; *C. R. Acad. Sc. 70* (1870) 1222-1226; reprinted in: KLEIN *Papers,* vol. I (paper VI), and LIE *Papers,* vol. I (paper III).

1870b «Über die Haupttangenten kurven der Kummerschen Fläche vierten Grades mit sechzehn Knotenpunkten»; *Sitzungsber. d. königl. preuss. Akad. d. Wiss. zu Berlin* 1870, 891-899; reprinted in: KLEIN *Papers,* vol. I (paper XXV), and LIE *Papers,* vol. I (paper X).

1871 «Über diejenigen Kurven, welche durch ein geschlossenes System von einfach unendlich vielen vertauschbaren linearen Transformationen in sich übergehen»; *Math. Ann. 4* (1871) 50-84; reprinted in: KLEIN *Papers,* vol. I (paper XXVI), and LIE *Papers,* vol. I (paper XIV).

KLEIN FELIX and A. SOMMERFELD

1897 *Über die Theorie des Kreisels,* 4 Bde. (B.G. Teubner, Leipzig 1897, 1898, 1903, 1910; reprinted in one volume in 1965 by Johnson Reprint Corp., New York and B.G. Teubner, Stuttgart).

KNESER, A.

1894 «Das Princip der kleinsten Action und die infinitesimale Transformation der dynamischen Probleme»; *Sitzungsber. d. Naturforscher-Gesellschaft bei der Universität Jurjew (Dorpat) 10,* Heft 3 (1894) 501-514.

1917 «Transformationsgruppen und Variationsrechnung»; *Journ. f. d. reine u. angew. Math. 147* (1917) 54-66.

1925 «Leopold Kronecker»; *Jahresber. d. deutschen Mathem.-Vereinig. 33* (1925) 210-228.

1928 *Das Prinzip der kleinsten Wirkung von Leibniz bis zur Gegenwart* (B.G. Teubner, Leipzig 1928).

KÖNIGSBERGER, LEO

1902 *Hermann von Helmholtz,* 3 vols. (Vieweg-Verlag, Braunschweig 1902, vol. I and 1903, vols. II + III).

1904 *Carl Gustav Jacob Jacobi* (B. G. Teubner, Leipzig 1904).

KOPPENFELS, WERNER V.

141 «Georg Prange»; *Jahresber. d. deutschen Mathem. Vereinig. 51* (1941) 1-14.

KORTUM, H.

1906 «Rudolf Lipschitz»; *Jahresbericht der deutschen Mathematiker-Vereinigung 15* (1906) 56-59.

KOSCHMIEDER, L.

1930 «Adolf Kneser»; *Sitzungsber. d. Berliner Mathem. Gesellschaft 29* (1930) 78-102.

KUCHARSKI, W.

1952 «Über Hamels Bedeutung für die Mechanik»; *Zeitschr. f. angew. Math. u. Mechanik 32* (1952) 293-297.

KUHN, THOMAS S.

1978 *Black-Body Theory and the Quantum Discontinuity 1894-1912* (Clarendon Press, Oxford and Oxford Univ. Press 1978).

KUMMER, ERNST EDUARD

Papers Collected Papers, ed. by A. Weil (Springer-Verlag, Berlin, Heidelberg, New York 1975): contains several obituaries and other articles on Kummer's work and life.

KUMPERA, A.

1975 «Invariants Différentiels d'un Pseudogroupe de Lie, I et II»; *Journ. Differential Geometry 10* (1975) 289-345, 347-416.

LAGRANGE, J.L. DE

1853 *Mécanique Analytique*, t. I, in: LAGRANGE *Papers*, vol. XI, publ. par les soins de Gaston Darboux. This is a reprint of the 3rd edition from 1853 by J. Bertrand, who added notes to Lagrange's own 2nd edition from 1811.

Papers *Oeuvres*, 14 vols. (Gauthier-Villars et fils, Paris 1867-1892).

LANDAU, L. D.

Papers *Collected Papers* ed. by D. ter Haar (Pergamon Press, Oxford London, Paris 1965).

LARMOR, JOSEPH

1900 *Aether and Matter* (Cambridge Univ. Press, Cambridge 1900).

LEBESGUE, HENRI

1923 «Notice sur la vie et les travaux de Camille Jordan (1838-1922)»; *Mémoires de l'Académie des Sciences* (2^e^ sér.) *58* (1923) 29-66; reprinted in: JORDAN *Papers*, vol. IV, p. X-XXIX.

LEBON, ERNEST

1912 *Henri Poincaré* 2^e^ ed. (Gauthier-Villars, Paris 1912).

1913 *Gaston Darboux*, 2^e^ ed. (Gauthier-Villars, Paris 1913).

LEJEUNE DIRICHLET, G.

1881 «Gedächtnisrede auf Carl Gustav Jacob Jacobi», in: JACOBI *Papers*, vol. I (1881) 1-21, and LEJEUNE DIRICHLET *Papers*, vol. II, 227-252.

Papers *Werke*, 2 vols. (G. Reimer, Berlin 1889-1897).

LEVI-CIVITA, TULLIO

1896 «Sulle trasformazioni delle equazioni dinamiche»; *Annali di Matem. Pura ed Appl.* (ser. II) *24* (1896) 255-300.

LÉVY, P.

1931 «R. Liouville (1856-1930)»; *Journ. de l'École Polytechnique* (2^e^ sér.) *29* (1931) 1-5.

LIE, SOPHUS

1872a «Über Komplexe, insbesondere Linien- und Kugelkomplexe, mit Anwendung auf die Theorie partieller Differentialgleichungen»; *Mathematische Annalen 5* (1872) 145-256; reprinted in: LIE *Papers*, vol. II, 1-121.

1872b «Zur Theorie der Differentialprobleme»; *Forhandlinger i Videnskabs-Selskabet i Christiania, Aar* 1872, 132-133; reprinted in: LIE *Papers*, vol. III, 27-28.

1874a «Zur Theorie des Integrabilitätsfaktors»; *Forhandlinger i Videnskabs-Selskabet i Christiania, Aar* 1874, 242-254; reprinted in: LIE *Papers*, vol. III, 176-187.

1874b «Begründung einer Invariantentheorie der Berührungstransformationen»; *Mathem. Ann. 8* (1874/1875) 215-303; reprinted in: LIE *Papers*, vol. IV (1929), 1-96.

1874c «Über Gruppen von Transformationen»; *Nachr. v. d. Königl. Ges. d. Wiss. z. Göttingen* 1874, 529-542; reprinted in: LIE *Papers*, vol. V (1924), 1-8.

1875 «Allgemeine Theorie der partiellen Differentialgleichungen erster Ordnung»; *Math. Ann. 9* (1875/1876) 245-296, *11* (1877) 464-557; reprinted in: LIE *Papers*, vol. IV (1929), 97-264.

1882 «Untersuchungen über geodätische Kurven»; *Mathem. Ann. 20* (1882) 357-454: reprinted in : LIE *Papers*, vol. II, 267-373.

1884 «Über Differentialinvarianten»; *Math. Ann. 24* (1884) 537-578; reprinted. in LIE *Papers*, vol. VI, 95-138.

1885 «Allgemeine Untersuchungen über Differentialgleichungen, die eine kontinuierliche, endliche Gruppe gestatten»; *Math. Ann. 25* (1885) 71-151; reprinted in LIE *Papers*, vol. VI, 139-223.

1891 «Die Grundlagen für die Theorie der unendlichen kontinuierlichen Transformationsgruppen, I und II»; *Ber. ü. d. Verhandl. d. königl. sächs. Ges. d. Wiss. zu Leipzig, Math. Phys. Kl. 43* (1891) 316-352, 353-393; reprinted in: S. LIE *Papers*, vol. VI, 300-330, 331-364.

1895 «Influence de Galois sur le développement des Mathématiques»; in: *Le Centenaire de l'Ecole Normale 1795-1895* (Librairie Hachette et C^ie^, Paris (1895) 481-489; reprinted in: LIE *Papers*, vol. VI, 592-601.

1897 «Die Theorie der Integralinvarianten ist ein Korollar der Theorie der Differentialinvarianten»; *Berichte über die Verhandlungen der königlich sächsischen Gesellschaft der Wissenschaften zu Leipzig, Mathem.-Phys. Kl. 49* (1897) 342-357; reprinted in LIE *Papers*, vol. VI, 649-663.

Papers *Gesammelte Abhandlungen*, 7 vols., herausgegeben von F. Engel und P. Heegaard (Teubner, Leipzig, and Aschehoug and Co., Kristiana or Oslo, 1922-1960).

LIE, SOPHUS and FRIEDRICH ENGEL

1888 *Theorie der Transformationsgruppen* (B. G. Teubner, Leipzig 1888-1893; reprinted by Chelsea Publ. Co., New York 1970).

LIE, SOPHUS and GEORG SCHEFFERS

1891 *Differentialgleichungen* (B. G. Teubner, Leipzig 1891; reprinted by Chelsea Publ. Co., New York 1967).

LIEBMANN, H.

1908 «Adolf Mayer»; *Jahresber, d. deutschen Mathem. Vereinig. 17* (1908) 355-362.

LIOUVILLE, RENÉ

1891 «Sur un problème d'Analyse qui se rattache aux équations de la Dynamique»; *C. R. Acad. Sc. 112* (1891) 710-712; *114* (1892) 974-977.

1892a «Sur les équations de la Dynamique»; *C. R. Acad. Sc. 114* (1892) 1171-1172.

1892b «Sur un problème qui se rattache aux équations de la Dynamique»; *C. R. Acad. Sc. 115* (1892) 403-406, 468.

1892c «Sur les équations de la Dynamique»; *C. R. Acad. Sc. 115* (1892) 646-648, 792-739.

1895 «Sur les équations de la Dynamique»; *Acta Mathematica 19* (1895) 251-283.

LIPSCHITZ, R.

1869 «Untersuchungen in Betreff der ganzen homogenen Functionen von n Differentialen»; *Journ. f. d. reine u. angew. Math. 70* (1869) 71-102.

1870a «Entwicklung einiger Eigenschaften der quadratischen Formen von n Differentialen»; *Journ. f. d. reine u. angew. Math. 71* (1870) 274-295.

1870b «Fortgesetzte Untersuchungen in Betreff der ganzen homogenen Functionen von n Differentialen»; *Journ. f. d. reine u. angew. Math. 72* (1870) 1-56.

1872 «Untersuchung eines Problems der Variationsrechnung, in welchem das Problem der Mechanik enthalten ist»; *Journ. f. d. reine u. angew. Math. 74* (1872) 116-149.

1877 «Bemerkungen zu dem Princip des kleinsten Zwanges»; *Journ. f. d. reine u. angew. Math. 82* (1877) 316-342.

LORENTZ, H.A.

1892 «La théorie électromagnétique de Maxwell et son application aux corps mouvants»; *Archives Néerlandaises des sciences exactes et naturelles 25* (1892) 363-452; reprinted in: LORENTZ *Papers*, vol. II, 164-249; appeared as a book (Leiden 1892), too.

1904 «Maxwells elektromagnetische Theorie», and: «Weiterbildung der Maxwellschen Theorie. Elektronentheorie»; in: *Encyklopädie der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen Bd. V,2* (B. G. Teubner, Leipzig 1904-1922) ch. V 13. (pp 63-144) and ch. V 14. (pp 145-280).

Papers *Collected Papers*, 9 vols. (M. Nijhoff, The Hague 1935-1939).

LOREY, W.

1921 «Paul Stäckel zum Gedächtnis»; *Zeitschr. f. mathem. u. naturw. Unterricht aller Schulgattungen 52* (1921) 85-88.

MARKOW, M.

1936 «Zur Diracschen Theorie des Elektrons, I. Teil»; *Physikal. Zeitschr. d. Sowjetunion 10* (1936) 773-808.

MAXWELL, JAMES CLERK

1873 *A Treatise on Electricity and Magnetism* 2 vols. (At the Clarendon Press, Oxford 1873).

MAYER, A.

1886 «Die beiden allgemeinen Sätze der Variationsrechnung, welche den beiden Formen des Princips der kleinsten Aktion in der Dynamik entsprechen»; *Berichte über die Verhandlungen der königlich sächsischen Gesellschaft der Wissenschaften zu Leipzig, Math.-Phys. Kl. 38* (1886) 343-355.

MEYER, W. FR.

1898 «Invariantentheorie»: in: *Encyklopädie der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen*, Bd. I, 1 (Teubner, Leipzig 1898-1904) 320-403.

MIE, GUSTAV

1912 «Grundlagen einer Theorie der Materie»; *Annalen der Physik* (4. Folge) *37* (1912) 511-534.

MILLER, ARTHUR I.

1981 *Albert Einstein's Special Theory of Relativity, Emergence (1905) and Early Interpretation (1905-1911)* (Addison-Wesley Publ. Co., Inc., Reading, Massachusetts 1981).

MINKOWSKI, HERMANN

1907 «Das Relativitätsprinzip» (lecture presented at a meeting of the Göttinger Mathematische Gesellschaft on Nov. 5, 1907), edited by A. Sommerfeld in: *Ann. d. Physik* (4. Folge) *47* (1915) 927-938.

1908a «Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern»; *Nachr. v. d. Königl. Ges. d. Wiss. zu Göttingen, Math.-phys. Kl.* 1908, 53-111; reprinted in: MINKOWSKI *Papers*, vol. II, 352-404.

1908b «Raum und Zeit», lecture at the 80th congress Deutscher Naturforscher und Ärzte at Cologne in Sept. 1908; published in: *Physikal. Zeitschr. 10* (1909) 104-111; reprinted in: MINKOWSKI *Papers*, vol. II, 431-444.

Papers Gesammelte Abhandlungen, 2 vols. (B. G. Teubner, Leipzig und Berlin 1911; reprinted by Chelsea Publ. Co., New York, N. Y. 1967).

MOSER, JÜRGEN

1973 *Stable and Random Motions in Dynamical Systems* (Princeton Univ. Press and Univ. of Tokyo Press, Princeton, N.J., 1973).

NOETHER, EMMY

1918a «Invarianten beliebiger Differentialausdrücke; *Nachr. von d. Königl. Gesellsch. d. Wiss. zu Göttingen, Math.-phys. Kl.* 1918, 37-44; reprinted in: NOETHER *Papers*, 240-247.

1918b «Invariante Variationsprobleme»; *Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen, Mathem.-physikalische Klasse*, 1918, 235-257. Reprinted as paper 13 in: NOETHER *Papers*, 248-270. English translation by M. A. Tavel in: *Transport Theory and Statistical Physics 1* (1971), 183-207.

Papers Collected Papers, ed. by N. Jacobson (Springer-Verlag, Berlin, Heidelberg, New York, Tokyo 1983).

NOETHER, MAX

1895 «Arthur Cayley»; *Mathem. Annalen 46* (1895) 463-480.

1898 «James Joseph Sylvester»; *Mathem. Annalen 50* (1898) 133-156.

1900 «Sophus Lie»; *Mathem. Annalen 53* (1900) 1-41.

1914 «Paul Gordan»; *Mathem. Annalen 75* (1914) 1-41.

OVSIANNIKOV, L.V.

1982 *Group Analysis of Differential Equations* (Academic Press, New York, London,...., 1982).

PAINLEVÉ, PAUL

1892a «Sur les transformations en Mécanique»; *C. R. Acad. Sc. 114* (1892) 901-904, 1104-1107, 1412-1414.

1892b «Sur les intégrales de la Dynamique»; *C. R. Acad. Sc. 114* (1892) 1168-1171.

1892c «Sur les transformations des équations de Lagrange»; *C. R. Acad. Sc. 115* (1892) 495-498.

1892d «Sur la transformation de équations de la Dynamique»; *C. R. Acad. Sc. 115* (1892) 714-717, 874-875

1893 «Sur les mouvements des systèmes dont les trajectoires admettent une transformation infinitésimale»; *C. R. Acad. Sc. 116* (1893) 21-24.

1894a «Sur les transformations infinitésimales des trajectoires des systèmes»; *C. R. Acad. Sc. 119* (1894) 637-639.

1894b «Mémoire sur la transformation des équations de la Dynamique»; *Journ. de Mathém. Pures et appl.* (4[e] sér.) *10* (1894) 5-92.

1895 *Leçons sur l'intégration des équations différentielles de la Mécanique et applications* (A. Hermann, Paris 1895).

1896 «Sur les transformations des équations de la Dynamique»; *C. R. Acad. Sc. 123* (1896) 392-395.

Papers Oeuvres de Paul Painlevé (Éditions du Centre National de la Recherche Scientifique, Paris 1975).

PAIS, ABRAHAM

1982 *«Subtle is the Lord...» The Science and the Life of Albert Einstein* (Oxford Univ. Press, Oxford and New York 1982).

PAULI, WOLFGANG

1921 «Relativitätstheorie»; in: *Encyklopädie der Math. Wissensch.*, (B G. Teubner, Leipzig 1904-1922), vol. V. 2 (1921) 539-775. English translation: *Theory of Relativity* (Pergamon Press, Oxford 1958). The article is reprinted in: PAULI *Papers*, vol I, 1-263.

1941 «Relativistic Field Theory of Elementary Particles»; *Rev. Mod. Physics 13* (1941) 203-232.

Papers Collected Scientific Papers, 2 vols., ed. by R. Kronig and V. F. Weisskopf (Interscience Publ., New York, London, Sydney 1964).

PERRON, OSCAR

1920 «Paul Stäckel»; *Sitzungsber. d. Heidelberger Akad. d. Wiss., Mathem.-naturw. Kl., Abt. A: Mathem.-physik. Wiss.* 1920, 7. Abhandlung, 1-20

PLANCK, MAX

1928 «Hendrik Antoon Lorentz»; *Die Naturwissenschaften 16* (1928) 549-555; reprinted in: PLANCK *Papers*, vol. III, 343-349.

Papers Physikalische Abhandlungen und Vorträge, 3 vols. (F. Vieweg und Sohn, Braunschweig 1958).

POGGENDORFF, J.C.

Biogr. Biographisch-Literarisches Handwörterbuch der Exakten Naturwissenschaften, 7 vols. (Akademie-Verlag, Berlin 1956).

POINCARÉ, HENRI

1890 «Sur le problème des trois corps et les équations de la dynamique»; *Acta Mathematica 13* (1890) 1-270; reprinted in: POINCARÉ *Papers*, vol. VII, 262-479.

1892 *Les Méthodes Nouvelles de la Mécanique Céleste*, 3 vols. (Gauthier-Villars et fils, Paris 1892-1899).

1900 «La théorie de Lorentz et le principe de réaction»; *Archives Néerland. des Sciences exactes et natur.* (2[e] sér.) 5 (1900) 252-279, ch.3; reprinted in: POINCARÉ *Papers*, vol. IX, 464-488.

1901a *Electricité et Optique*, 2[e] ed. (Gauthier-Villars, Paris 1901).

1901b «Sur une forme nouvelle des équations de la Mécanique»; *C. R. Acad. Sc. 132* (1901) 369-371; reprinted in: POINCARÉ *Papers*, vol. VII, 218-219.

1906 «Sur la dynamique de l'électron»; *Rendiconti del Circolo Matematico di Palermo 21* (1906) 129-176; reprinted in: POINCARÉ *Papers* vol. IX (Gauthier-Villars, Paris 1954) 494-550.

Papers Oeuvres, 10 vols. (Gauthiers-Villars, Paris 1928-1954).

PRANGE, GEORG

1904 «Die allgemeinen Integrationsmethoden der analytischen Mechanik»; in: *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen*, Bd. IV, 2 (B. G. Teubner, Leipzig 1904-1935), 505-804.

REID, CONSTANCE

1970 *Hilbert* (Springer-Verlag, Berlin, Heidelberg, New York, 1970).

1976 *Courant in Göttingen and New York. The Story of an Improbable Mathematician* (Springer-Verlag, New York, Heidelberg, Berlin, 1976).

RIEMANN, BERNHARD

1854 «Über die Hypothesen, welche der Geometrie zu Grunde liegen» (inaugural lecture from 1854): RIEMANN *Papers*, 272-287.

1861 «Commentatio mathematica, que respondere tentatur questioni ab Illustrissima Academia Parisiensi propositae; in: RIEMANN *Papers*, 391-423 (submitted in 1861 to the Académie des Sciences in Paris).

Papers Bernhard Riemann's gesammelte mathematische Werke und wissenschaftlicher Nachlass, 2. Aufl., bearbeitet von H. Weber (Teubner, Leipzig 1892; reprinted by Dover Publications, Inc., New York 1953).

ROMAN, PAUL

1960 *Theory of Elementary Particles* (North Holland, Amsterdam 1960) 217-220.

ROSENFELD, LÉON

1940 «Sur le tenseur d'impulsion-énergie»; *Mémoires Acad. Roy. de Belgique, Cl. Sc. 18* (1940), no. 6, 1-30. English translation in: ROSENFELD *Papers*, 711-735.

Papers Selected Papers, ed. by R. S. Cohen and J. J. Stachel (Boston Studies in the Philosophy of Science 21, D. Reidel Publ. Co., Dordrecht, Boston, London, 1979).

ROUTH, E.J.

1877 *Stability of a given state of motion* (Macmillan and Co., London 1877); reprinted in: *Stability of Motion*, ed. by A. T. Fuller (Taylor and Francis Ltd., London 1975).

RUMER, G.

1931 «Der gegenwärtige Stand der Diracschen Theorie des Elektrons»; *Physikal. Zeitschr. 32* (1931) 601-622.

1973 «Zum Gedenken an L. D. Landau», *Physik. Blätter 29* (1973) 147-152.

RZEWUSKI, JAN
1953 «Differential Conservation Laws in Non-Local Field Theories»; *Il Nuov. Cim. 10* (1953) 784-802.
1958 *Field Theory, vol. I: Classical Theory* (PWN-Polish Scient. Publ., Warsaw 1958).
SARTON, G.
1944 «Lagrange's personality»; *Proceed. of the Amer. Philos. Soc., 88* (1944) 457-496.
SCHMEIDLER, WERNER
1952 «Bericht über die Hamel-Feier anlässlich des Salzburger Mathematiker-Kongresses 1952»; *Sitzungsber. d. Berliner Mathem. Ges.* 1951/52 (1952) 27-34.
1955 «Zum Gedächtnis an Georg Hamel»; *Jahresber. d. deutschen Mathem. Vereinig. 58* (1955) 1-5.
SCHUR, FRIEDRICH
1931 «Nachruf auf Otto Staude»; *Jahresber. d. deutschen Mathem.-Vereinig. 40* (1931) 219-223.
SCHUR, ISSAI
1928 *Vorlesungen über Invariantentheorie,* bearb. und herausgeg. von Helmut Grunsky (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen 143; Springer-Verlag, Berlin, Heidelberg, New York, 1968) (Schur's lectures from 1928).
SCHÜTZ, J.R.
1897 «Prinzip der absoluten Erhaltung der Energie»; *Nachr. v. d. königl. Ges. d. Wiss. zu Göttingen, Mathem.-phys. Kl.* 1897, 110-123.
SCHWARZSCHILD, K.
1903 «Zur Elektrodynamik. I: Zwei Formen des Princips der kleinsten Action in der Electronentheorie»; *Nachr. v. d. Königl. Ges. d. Wiss. zu Göttingen, Math.-phys. Kl.* 1903, 126-131.
SCHWEBER, SILVAN S., HANS A. BETHE and FREDERIC DE HOFFMANN
1955 *Meson and Fields I: Fields* (Row, Peterson and Co., Evanstan, Ill. and White Plains, N.Y., 1955).
SCHWINGER, JULIAN
1951 «The Theory of Quantized Fields I»: *Phys. Rev. 82* (1951) 914-927.
SEGRE, B.
1954 «Commemorazione del Socio Guido Fubini», *Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat.* (ser. 8) *17* (1954) 276-294; reprinted in: FUBINI *Papers*, vol. I, 14-29.
SOMMERFELD, ARNOLD
1916 «Karl Schwarzschild»; *Die Naturwissenschaften 4* (1916) 453-457.
SRINIVASAN, B. *et al.* (eds.)
1983 *Emmy Noether in Bryn Mawr;* Proceedings of a symposium sponsored by the Association for Women in Mathematics in honor of Emmy Noether's 100th birthday (Springer-Verlag, Berlin 1983).
STÄCKEL, PAUL
1891 «Über die Differentialgleichungen der Dynamik und den Begriff der analytischen Aequivalenz dynamischer Probleme»; *Journ. f. d. reine u. angew. Math. 107* (1891) 319-348.
1893a «Über die Bewegung eines Punktes in einer n-fachen Mannigfaltigkeit; *Math. Ann. 42* (1893) 537-563.
1893b «Über dynamische Probleme, deren Differentialgleichungen eine infinitesimale Transformation gestatten»; *Ber. ü. d. Verh. d. Königl. sächs. Ges. d. Wiss. zu Leipzig, Math.-Phys. Kl* 45, (1893) 331-340.
1894a «Sur des problèmes de Dynamique dont les équations différentielles admettent une transformation infinitésimale»; *C. R. Acad. Sc. 119* (1894) 508-510.
1894b «Sur des problèmes de Dynamique dont les équations différentielles admettent un groupe continu; *C. R. Acad Sc. 119* (1894) 723-725.
1897 «Anwendungen von Lie's Theorie der Transformationsgruppen auf die Differentialgleichungen der Dynamik»; *Ber. ü. d. Verh. d. Königl. sächs. Ges d. Wiss zu Leipzig, Math.-Phys. Kl. 49* (1897) 411-442 (this is an extended summary of the previous work of Painlevé, Staude and Stäckel himself).
1898 «Über Transformationen von Bewegungen»; *Nachr. v. d. königl. Ges. d. Wiss. zu Göttingen, Math.-Pnys. Kl.* 1898, 157-165.
1901 «Elementare Dynamik der Punktsysteme und starren Körper»; in: *Encyclopädie der Mathem. Wiss. mit Einschluss ihrer Anwendungen,* Bd. IV, 1 (B. G. Teubner, Leipzig 1901-1908), Artikel 6.

STAUDE, OTTO
1892 «Über die Bahncurven eines auf einer Oberfläche beweglichen Punktes, welche infinitesimale Transformationen zulassen»; *Ber. über d. Verh. d. königl. sächs. Ges. d. Wiss. zu Leipzig, Math.-Phys. Kl. 44* (1892) 429-446.
1893a «Über die Bahncurven eines in einem Raume von drei Dimensionen beweglichen Punktes, welche infinitesimale Transformationen zulassen»; *Ber. ü. d. Verh. d. königl. sächs. Ges. d. Wiss. zu Leipzig, Math.-Phys. Kl. 45* (1893) 511-522.
1893b «Über die Verallgemeinerung eines Satzes aus der Theorie der Centralbewegungen»; *Sitzungsberichte der Naturforscher-Gesellschaft bei der Universität Jurjew (Dorpat) 10,* Heft 2 (1893) 328-357.
SYLVESTER, J.J.
1851 «On the general theory of associated algebraical forms»; *Cambridge and Dublin Mathematical Journal 6* (1851) 289-293; reprinted in: SYLVESTER *Papers*, vol. I, 198-202.
1852 «On the principles of the calculus of forms»; *Cambridge and Dublin Mathematical Journal 7* (1852) 52-97, 179-217; reprinted in: SYLVESTER *Papers*, vol. I, 284-363
Papers Collected Mathematical Papers, 4 vols. (Cambridge Univ. Press, Cambridge 1904-1912; reprinted by Chelsea Publishing Company, New York, N.Y. 1973).
TIETZE, HEINRICH
1954 «Gustav Herglotz»; *Jahrbuch der Bayerischen Akademie der Wissenschaften 1953* (München 1954) 188-194.
TRAUTMAN, ANDRZEJ
1956a «On the Conservation Theorems and Equations of Motion in Covariant Field Theories»; *Bull. de l'Acad. Polon. des Sc.* (Cl. III) *4* (1956) 675-678.
1956b «Killing's Equations and Conservation Theorems»; *Bull. Acad. Polon. Sc.* (Cl. III) *4* (1956) 679-682.
1957 «On the Conservation Theorems and Co-ordinate Systems in General Relativity»; *Bull. Acad. Polon. Sc.* (Cl. IIII) *5* (1957) 721-727.
1962 «Conservation Laws in General Relativity»; in: *Gravitation: an introduction to current research,* ed. by Louis Witten (J. Wiley and Sons, Inc., New York, London, 1962) 169-189.
1965 «Foundations and Current Problems of General Relativity»; in: *Lectures on General Relativity, Proceed. of the 1964 Brandeis Univ. Summer Inst. in Theor. Physics,* vol. I (Prentice Hall, Inc., Englewood Cliffs, N. J., 1965) 1-248.
1967 «Noether Equations and Conservation Laws»; *Commun. Math. Physics 6* (1967) 248-261.
VAN DER WAERDEN, B. L.
1932 *Die gruppentheoretische Methode in der Quantenmechanik* (Die Grundlehren der math. Wiss. in Einzeldarstellungen 36; J. Springer, Berlin 1932).
1935 «Nachruf auf Emmy Noether»; *Mathem. Annalen 111* (1935), 469-476; English translations in DICK 1981, 100-111, and in BREWER and SMITH 1981, 93-98.
1955 *Algebra I, 4. Aufl.* (Springer-Verlag, Berlin, Göttingen, Heidelberg, 1955).
VERMEIL, HERMANN
1918 «Bestimmung einer quadratischen Differentialform aus der Riemannschen und den Christoffelschen Differentialinvarianten mit Hilfe von Normalkoordinaten»; *Mathem. Ann. 79* (1918) 289-312.
VESSIOT, E.
1899 «Gewöhnliche Differentialgleichungen; Elementare Integrationsmethoden», in: *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen,* Bd. II, 1A (B. G. Teubner, Leipzig 1899-1916) 230-293.
VON WEBER, EDUARD
1899 «Partielle Differentialgleichungen»; in: *Encyckl. d. math. Wiss. II,* 1A, 294-399 (B.G. Teubner, Leipzig 1899-1916) 294-399.
VOSS, A.
1901 «Die Prinzipien der rationellen Mechanik»; in: *Encyclopädie der mathem. Wiss. mit Einschluss ihrer Anwendungen,* Bd. IV, 1 (B.C. Teubner, Leipzig 1901-1903), Artikel 1.
1918 «Gaston Darboux»; *Jahresber. d. deutschen Mathem.-Vereining, 27* (1918) 196-217.
WEBER, H.
1893 «Leopold Kronecker»; *Math. Ann. 43* (1893) 1-25.
WEITZENBÖCK, ROLAND
1923 *Invariantentheorie* (P. Noordhoff, Groningen 1923).

1927 «Neuere Arbeiten der algebraischen Invariantentheorie. Differentialinvarianten»; in: *Encyklopädie der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen*, Bd. III, 3 (Teubner, Leipzig 1902-1927), ch. 10 (denoted III D 10) 1-71.
WENTZEL, GREGOR
1943 *Einführung in die Quantentheorie der Wellenfelder* (Fr. Deuticke, Wien 1943).
WEYL, HERMANN
1922 *Space-Time-Matter*, 4th ed. (Dover Publications, Inc., New York 1950; this is a reprint of the English translation of the 4th German edition (1921), publ. in 1922 by Methuen, London)
1928a «Quantenmechanik und Gruppentheorie»; *Zeitschr. f. Physik 46* (1928) 1-46; reprinted in: WEYL *Papers*, vol. III, 90-135.
1928b *Gruppentheorie und Quantenmechanik* (S. Hirzel, Leipzig 1928, 2. Auflage 1931).
1930 «Felix Kleins Stellung in der mathematische Gegenwart»; *Die Naturwissenschaften 18* (1930) 4-11; reprinted in: WEYL *Papers*, vol. III, 292-299.
1935 «Emmy Noether»; *Scripta mathematica 3* (1935) 201-220; reprinted in: WEYL *Papers*, vol. III, 425-444 and in DICK 1981, 112-152.
1939 «Invariants»; *Duke Mathem. Journal 5* (1939) 489-502; reprinted in: WEYL *Papers*, vol. III, 670-683.
1944 «David Hilbert and his mathematical work»; *Bulletin of the Amer. Mathem. Soc. 50* (1944) 612-654; reprinted in: WEYL *Papers*, vol. IV, 130-172.
1946 *The Classical Groups, their Invariants and Representations*, 2nd Ed., (Princeton Univ. Press, Princeton, N. J., 1946).
Papers *Gesammelte Abhandlungen*, 4 vols. ed. by K. Chandrasekharan (Springer-Verlag, Berlin, Heidelberg, New York, 1968).
WHITEMAN, K.J.
1977 «Invariants and stability in classical mechanics»; *Reports on Progress in Physics 40* (1977) 1033-1069.
WIGNER, EUGENE P.
1927a «Über nicht kombinierende Terme in der neueren Quantentheorie. II. Teil»; *Zeitschr. f. Physik 40* (1927) 883-892.
1927b «Einige Folgerungen aus der Schrödingerschen Theorie für die Termstrukturen»; *Zeitschr. f. Physik 43* (1927) 624-652.
1931 *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren* (Die Wissenschaft, Bd. 85; Fr. Vieweg and Sohn, Braunschweig 1931).
1949 «Invariance in Physical Theory»; *Proceed. of the American Philosoph. Soc. 93* (1949) 521-526; reprinted in WIGNER 1967, 3-13.
1954 «Conservation laws in Classical and Quantum Physics»; *Progress of Theor. Physics 11* (1954) 437-440.
1963 «Events, Laws of Nature, and Invariance Principles»; in: *Nobel Lectures, Physics*, 1963-1970 (Elsevier Publ. Co., Amsterdam, London, New York 1972) 6-17, reprinted in: WIGNER 1967, 38-50.
1964 «Symmetry and Conservation Laws»; *Proceed. Nat. Acad. Sciences 51* (1964) 956-965, reprinted in: *Physics Today 17* (1964) 34-40 and in: WIGNER 1967, 14-27.
1967 *Symmetries and Reflections* (Indiana Univ. Press, Bloomington and London 1967).
WINTERNITZ, PAVEL
1983 «Lie Groups and Solutions of Nonlinear Differential Equations»; in: *Nonlinear Phenomena*, ed. by K. B. Wolf (Lecture Notes in Physics 189; Springer-Verlag, Berlin, Heidelberg, New York, Tokyo 1983) 263-331.
WINTNER, AUREL
1941 *The Analytical Foundations of Celestial Mechanics* (Princeton Mathem. Series 5, Princeton Univ. Press, Princeton, N. J., 1941).

DISCUSSION

BERGIA:

I have two questions. The first one is on a rather minor point. I have recently come across a paper by Levi-Civita of 1917[1] in which he makes use, if I remember correctly, of the Bianchi identities. It seems to me, but I might be wrong, that Bianchi's work on the subject goes back to the beginning of the century. Is that right?

KASTRUP:

That might be[2].

BERGIA:

Second point: you mentioned the notion of Killing vectors in relation to Lie's work. In the recent literature on differential geometry we are accustomed to Lie derivatives, Killing vectors and so on and so forth. Can you tell us whether these concepts can really be found in Lie's work? Or are they a recent attribution to him?

KASTRUP:

They can be found in Lie's work. There is no doubt about that[3].

BERGIA:

And the Killing vector in this context was a vector field such that the Lie derivative of the metric tensor along its integral curves vanishes...

KASTRUP:

Yes, this is contained in Lie's work, without of course, the name «Killing vector»! Lie was later quite critical of Killing, because he had the impression that Killing did not give him credit enough. Lie highly appreciated E. Cartan who gave him credit. During the later years of his life Lie became very touchy concerning priorities and the credits (to his work) he expected from other authors[4].

BACRY:

Just two minor points. The first one is just to mention that there exist many versions of Klein's *Erlanger Programm*. There are some which are more complete than other ones. I know that the one in the last French edition is complete.

A second point is about Lie groups. I think that what we are thinking when we speak of a Lie group today is not exactly what it was at that time, even just before the second world war. Not only because Lie groups were at that time considered as infinitesimal Lie groups, as local groups, but also because if you had a given abstract group acting in different ways on two manifolds, it was considered at that time as two *different* groups.

KASTRUP:

Of course, the main emphasis at that time was on infinitesimal transformations. Lie did discuss the integration of 1-dimensional Lie algebras, but I agree, Lie's main interest was in infinitesimal transformations. He did not study the global structure of group manifolds, etc...

1. T. LEVI-CIVITA, *Rend. Acc. Lincei*, series 5.°, vol. 26, 381 (1917).
2. See, e.g., M. BIANCHI, *Lezioni di geometria differenziale*, vol. 1, Spoerri/Pisa, 1902.
3. See, e.g., S. LIE and F. ENGEL, *Theorie der Transformationsgruppen*, Bd. I, Leipzig 1888.
4. As to Lie's opinion about Killing and Cartan see: S. LIE and F. ENGEL, *Theorie der Transformationsgruppen*, Bd. III, Leipzig 1893, § 142.

HERMANN:

As to the different versions of the *Erlanger Programm,* I suspect the following: An extension of the original version came when Klein himself edited his collected papers. There he added long footnotes and comments. Probably the original version was translated into French. I would expect an explanation as simple as this.

SPEISER:

I just would like to ask a question. There is the famous Helmholtz-Lie theorem which, I was always told, was first discovered by Helmholtz and then put in a correct form by Lie. I would like to know, whether there was here too a relation to Felix Klein.

KASTRUP:

Klein had his hands in that «affair», too. He had seen the paper by Helmholtz —Klein was very much aware of what was going on, Lie was not. He asked Lie to apply his theory of groups to this space problem of Helmholtz[5].

SPEISER:

That is important. Because, as you know, the Helmholtz-Lie theorem contains not only a very interesting idea and, I think, fundamental in its own right. It was also the starting point for Weyl's investigations, on what he called the space problem and was the subject of his Barcelona lectures.

KASTRUP:

Yes, Klein drew Lie's attention to Helmholtz' paper.

SPEISER:

Thank you very much. May I ask another small question. You mentioned the work of Emmy Noether on these differential invariants. Had she had contact at that time with Finsler? Was there a relation between Finsler and Noether?

KASTRUP:

I don't think so. Finsler was in Göttingen much later, as far as I remember![6].

SPEISER:

Not so much, a few years only.

KASTRUP:

But she never mentions Finsler in her papers.

SPEISER:

Of course, but it could have been the other way round. Did she influence Finsler?

KASTRUP:

I don't know. I cannot answer this question.

SPEISER:

Thank you very much.

5. Lie's papers on this problem are contained in vol. II of his collected works, papers V-VIII. See also the editorial notes on these papers in that volume.
6. Here my memory was incorrect: Paul Finsler (1894-1970) received his Ph.D. in 1918 in Göttingen, where Carathéodory was his thesis adviser. In 1922 he went to Cologne (as *Privatdozent*) and do Zürich in 1927 (as Professor).

PAIS:

Can I come back to your remark about Herglotz' paper of 1911? You said that it led Klein to ask Engel to do the same for the Galileo group.

KASTRUP:

Yes, that is true.

PAIS:

That I did not know. Did Engel actually do that? Was it published?

KASTRUP:

Yes he did. His work was published in the proceedings of the Göttingen Academy of Sciences (I have copies with me, I can show you the papers).

PAIS:

Thank you. I have a second part to my question. What did Engel then do? I can imagine what he did but how does that relate to the variation of $\int L dt$ for a little thing? Was Engel's contribution novel?

KASTRUP:

I think it was. Engel took the full 10-parameter Galilean group and then applied Lie's theory. He uses the invariant $p_j\, dq^j - H\, dt$ which, of course, is equal to $L dt$, but he never mentions the action integral. He derived all the ten conservation laws associated with n-body problems in mechanics. I think this was a new contribution which is mainly forgotten. It has been quoted, however, several times by Prof. Wigner[7].

PAIS:

Thanks very much.

WIGNER:

We have learned so much from your talk! But could you tell us who realized first that invariance does lead to the conservation laws? Usually Emmy Noether is quoted, but we all know that she was preceded by others. But who was the first one? Or is it a difficult question?

KASTRUP:

It is not an easy question, no. You never ask easy questions [laughs][8].

WIGNER:

Danke. Poincaré's work which you mentioned and which I did not know about, was later?

KASTRUP:

No. Poincaré's work was around 1890, but he did not use the action integral. He used his notion of integral invariants.

7. See footnote 150 of my preceding lecture (page 139).
8. This was the only correct part of my answer during the conference, where I pointed out the work of Stäckel and Kneser —see its section 6 above— in my answer to Wigner's question. Only after the conference, when I prepared my written contribution, did I realize that Lagrange and Hamilton are the names I should have mentioned in my answer, as can be seen from its section 6 above.

WIGNER:

But if he used the integral invariants and derived from them in 1890 the conservation laws...

KASTRUP:

Poincaré discussed the 10 classical conservation laws in connection with the symplectic form $dp_j \wedge dq^j$ [9].

WIGNER:

Thank you very much. Poincaré did something very relevant, but, apparently Stäckel did the final work!

KASTRUP:

I would say, the final work was really done by Emmy Noether. The most general theory was written down by her.

WIGNER:

Hamel has the conservation laws, in his book much before that but... [laughs].

MILLER:

A comment and a question. My comment is that I was under the impression (and this seems to be born out in the correspondence I have studied between Lie and Poincaré) that Lie's initial motivation for studying infinitesimal transformations was to elucidate and derive in a more precise manner than had been done before, the Riemannian line element in any number of dimensions —something which Helmholtz had attempted to do, and on which in a letter to Poincaré, Lie mentions how clumsy a mathematician Helmholtz was. You could say that Lie's real motivation was to solve differential equations indeed...

KASTRUP:

Lie's work on Helmholtz' space problem was later than his main work on groups. To solve differential equations was the first motive he had, at least he said so later.

MILLER:

But I am talking about the way the discovery came about. The initial motivation was the study of line elements or geodesics —as one might call them— and then he moved into differential equations.

KASTRUP:

Well, I do not think so. You can find everything in the collected papers of Lie, which are contained in 7 volumes, with notes and comments by the editors Engel and Heegard, including many letters by Lie, Klein, Mayer and others.

MILLER:

I am talking on the basis of a particular letter that Lie sent to Poincaré in 1882, about his motivation for discovering continuous groups. I just thought I should mention that to you.

Secondly, I wonder if you could comment or mention when Noether's theorem was explicitly introduced into quantum mechanics. The fact that conservation laws are connected with the symmetry of a system, I believe, was first mentioned in a paper by Prof. Wigner and von Neumann in 1926. But the name Noether is not in there.

9. As to Poincaré's work see section 6 of my preceding lecture.

KASTRUP:

No. Noether did not play a role, as far as I know, in quantum mechanics. Symmetries in quantum mechanics were introduced, as you mentioned already, by Prof. Wigner and von Neumann[10].

WIGNER:

I think the connection between the invariances and the conservation laws is obvious in quantum mechanics. As a matter of fact, when I teach this in classical mechanics I always explain that it is easier to derive them by first using quantum mechanics and then telling that classical mechanics is really an approximation to quantum mechanics.

MICHEL:

I would also like to ask Prof. Wigner to comment on the role of Hamel. Because Prof. Wigner said that he knew very well Hamel's work, and it seemed that it was important.

WIGNER:

I read Hamel's book. Hamel was professor of physics at the Institute of Technology in Berlin when I studied. His book I think was published in 1901 but I would not swear to that. It contained the derivation of the conservation laws from the invariances. So that struck me then.

KASTRUP:

That book is a little bit later. The habilitation thesis of Hamel was published in the *Mathematischen Annalen,* and also in the *Zeitschrift für Physik und Mathematik* in 1905. But the book is later, as far as I recall[11].

WIGNER:

I am sorry because I apparently made a mistake. But it was much before Noether!

SPEISER:

Just one more question, since you are such a *Fundgrube*. There was much work done in the middle of the last century by Weierstrass on the calculus of variations. I am not familiar with it, but isn't there anything which applies to mechanics and to this kind of problems, if not explicitly at least in an obvious transparent way?

KASTRUP:

Not in the work of Weierstrass, as far as I know.

SPEISER:

But then it went somehow into Hilbert's works.

KASTRUP:

Yes. The independent integral of Hilbert had some connection with Weierstrass' work. That is true.

SPEISER:

Thank you very much.

10. See footnote 162 of my preceding lecture (page 141).
11. HAMEL's book *Elementare Mechanik* appeared in 1912 (Teubner, Leipzig). It does not contain any discussion between symmetries and conservation laws. In ch. IV of that book Hamel derives the classical conservation laws from the equations of motion, without referring to symmetries. As to Hamel's application of group theory to mechanics see section 6 of my preceding lecture.