Multiloop Feynman diagrams: from approximate to exact results

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- Introduction
- Harmonic functions and the level structure of basis
- Single-Scale diagrams, PSLQ
- Integrals on the lattice
- Conclusion

F. D. belong to the class of (multiple) hypergeometric series

$$F\left(\begin{matrix} a_1 + b_1\epsilon, \ldots \\ A_1 + B_1\epsilon, \ldots \end{matrix} \middle| x, y, z \right)$$

kinematical variables

- The problem of expansion in ϵ. What functions are relevant here?

- ~~Ka~~ If there is a "level (weight)" structure of basic functions

$$f_a(x, \ldots)$$

weight index $a = 1, 2, 3, \ldots$

"a" is related to number of loops or depth of ϵ- expansion.

- Relation between thresholds or topology and the basis functio[n]

Toy example:

Given the first 3 coefficients of expansion

$$I = z + \frac{1}{2} z^2 + \frac{1}{3} z^3 + \ldots$$

Q: what is the rest of the series and what is the function I?

A: Make a "clever" guess

$$I = \sum z^n c_n \qquad c_n = \frac{1}{n} \; (?)$$

$$\Rightarrow \quad I = -\log(1-z)$$

Q: How to make clever guesses in general?

A:

- Suppose there exist set of functions
  $$\varphi_j(n) \quad , \; n\text{-integer}$$
  that form the <u>basis</u>
- Then make Ansatz
  $$c_n = x_1 \varphi_1(n) + x_2 \varphi_2(n) + \ldots + x_r \varphi_r(n)$$
  and solve the system of linear equations for $x_1, \ldots, x_r$.

example:

$$z = \frac{q^2}{m^2}$$

$$q^2 I = \sum_{n>0} c_n z^n$$

First few coefficients are found explicitely by large mass expansion

$$c_0 = 0, \quad c_1 = 2\zeta_2, \quad c_3 = \zeta_2 + \frac{1}{2}, \quad c_4 = \frac{2}{3}\zeta_2 + \frac{1}{2},$$

$$c_5 = \frac{1}{2}\zeta_2 + \frac{65}{144}, \quad \text{etc.}$$

Our Ansatz:

$$c_n = \frac{\zeta_2}{n} x_1 + \zeta_2 S_1 x_2 + S_3 x_3 + S_2 S_1 x_4 + S_1^3 x_5 + \frac{S_2}{n} x_6 + \frac{S_1^2}{n} x_7 + \frac{S_1}{n^2} x_8 + \frac{1}{n^3} x_9$$

where $S_a = \sum_{j=1}^{n-1} \frac{1}{j^a}$ – harmonic sum of weight a

Result:

$$q^2 I = \sum z^n \left[2\frac{\zeta_2}{n} + 2\frac{S_2}{n} - 2\frac{S_1}{n^2} \right]$$

Or, summing up

$$= -2\zeta_2 \log(1-z) - 2\log(1-z)\,\mathrm{Li}_2(z) - 6 S_{1,2}(z)$$

General arguments:

A Feynman diagram to be written as

$$J(z) = \sum_{j_1 \ldots j_s} \frac{\Gamma(l_1\{j,d\}) \ldots \Gamma(l_\nu\{j,d\})}{\Gamma(L_1\{j,d\}) \ldots \Gamma(L_\mu\{j,d\})} z^{\lambda(\{j,d\})}$$

Expansion of Γ-function in $\varepsilon = \frac{d-4}{2}$

$$\frac{\Gamma(j+a\varepsilon)}{\Gamma(1+a\varepsilon)} = (j-1)! \exp\left\{-\sum \frac{(-a\varepsilon)^k}{k} S_k(j-1)\right\} =$$

$$= (j-1)! \left[1 + a\varepsilon S_1 + \frac{1}{2} a^2 \varepsilon^2 (S_2 - S_1^2) + \ldots\right]$$

This suggests the recipe of generalization to higher lever functions.

- Let $\varphi_a(n) = \sum_{\{j\}} c_{\{j\}}$ be basis of weight a
- then

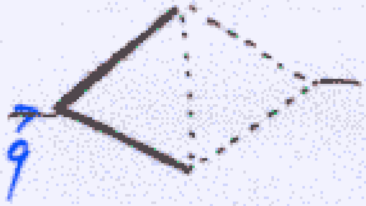
$$\varphi_{a+1}(n) = \sum c_{\{j\}} S_1(\{j\})$$

$$\varphi_{a+2}(n) = \begin{cases} \sum c_{\{j\}} S_2(\{j\}) \\ \sum c_{\{j\}} S_1(\{j\}) S_1(\{j\}) \end{cases}$$

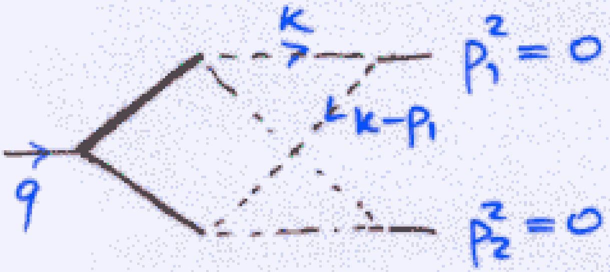
i.e. insert harmonic sums in/out of summation $\sum c_{\{j\}}$.

Structure of "harmonic" basis.
(Relevant when $z_{th} = 1$)

weight (level)	
5	$\cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot$
4	$\frac{1}{n^4} \quad \frac{1}{n^3}S_1 \quad \frac{1}{n^2}S_2 \quad \cdots\cdots \quad \frac{1}{n^2}\zeta_2 \quad \frac{1}{n}\zeta_3 \quad \zeta_4$
3	$\frac{1}{n^3} \quad \frac{1}{n^2}S_1 \quad \frac{1}{n}S_2 \quad \frac{1}{n}S_1^2 \quad \frac{1}{n}\zeta_2 \quad \zeta_3$
2	$\frac{1}{n^2} \quad \frac{1}{n}\cdot S_1 \quad \zeta_2$
1	$\frac{1}{n}$



$= \; ?$ $\qquad z = \dfrac{q^2}{m^2}$



$\sim \dfrac{1}{\epsilon^2}$ (double collinear divergence)

$$= \int_0^1 d\alpha \cdot \int \frac{dk}{(k^2 - 2\alpha k p_1)^2} \cdot \Phi = \int_0^1 d\alpha \int \frac{dk}{(k - \alpha p_1)^4} \cdot \Phi$$

To compute the highest pole it is enough to replace

$$\frac{1}{(k - \alpha p_1)^4} \longrightarrow i \frac{\pi^2}{\epsilon} \delta(k - \alpha p_1)$$

and the same trick for other two lines.

$$I = -\frac{1}{\epsilon^2} \frac{1}{(q^2)^2} \int_0^1 \frac{d\alpha_1 \cdot d\alpha_2}{[1 - \alpha_1(1-\alpha_2) \cdot z][1 - (1-\alpha_1)\alpha_2 z]}$$

$$= -\frac{1}{\epsilon^2} \frac{1}{(q^2)^2} \sum_{n=1}^{\infty} z^n \frac{1}{\binom{2n}{n}} 6 \sum_{j=1}^{n-1} \binom{2j}{j} \frac{1}{j}$$

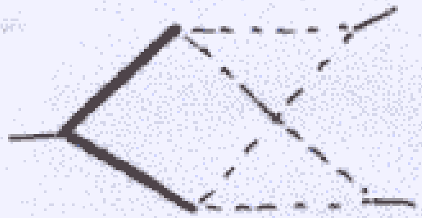
New structure:

$$\boxed{W_a(n) = \sum_{j=1}^{n} \binom{2j}{j} \frac{1}{j^a}}$$

Structure of "binomial" basis
(Relevant when $z_{th} = 4$)

weight (level)					
4	$\cdot\ \cdot\ \cdot$	$\cdot\ \cdot\ \cdot$	$\cdot\ \cdot\ \cdot$	$\cdot\ \cdot\ \cdot$	$\cdot\ \cdot\ \cdot$
3	$\frac{1}{\binom{2n}{n}} \cdot W_3$	$\frac{1}{\binom{2n}{n}} \frac{1}{n} W_2$	$\frac{1}{\binom{2n}{n}} \frac{1}{n} W_1 \cdot S_1$	$\frac{1}{\binom{2n}{n}} \frac{1}{n^2} W_1$	$\frac{1}{\binom{2n}{n}} W_1 \zeta_2$
2		$\frac{1}{\binom{2n}{n}} W_2$	$\frac{1}{\binom{2n}{n}} \frac{1}{n} W_1$	$\frac{1}{\binom{2n}{n}} W_1 \cdot S_1$	
1			$\frac{1}{\binom{2n}{n}} W_1$		

Result:

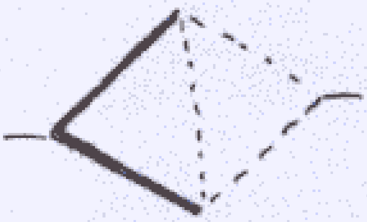


$$= \frac{1}{(q^2)^2} \sum z^n \frac{1}{\binom{2n}{n}} \cdot \frac{1}{n}$$

$$\Big\{ -\frac{6}{\epsilon^2} W_1 + \frac{1}{\epsilon}\left[-4W_2 - 12 W_{1,1} - 12 S_1 W_1 - 16 S_2\right]$$

$$-18 \zeta_2 W_1 - 4W_3 - 8S_3 + 12 W_{1,2} - 8 W_{2,1} - 12 W_{1,(1+1)}$$

$$-16 S_{1,2} - 8 S_1 W_2 - 24 S_1 W_{1,1} - 16 S_1 S_2 - 12 S_1^2 W_1 \Big\}$$



$$= \frac{1}{q^2} \sum z^n \frac{1}{n^2}$$

$$\left\{ \frac{1}{n} + \frac{1}{\binom{2n}{n}} \left(-2 \log(-z) - 3 W_1 + \frac{2}{n} \right) \right\}$$

$$= \frac{1}{q^2} \Big\{ \mathrm{Li}_3(z) - 6\zeta_3 - \zeta_2 \log y - \frac{1}{6} \log^3 y - 4 \log y \, \mathrm{Li}_2(y)$$

$$+ 4 \mathrm{Li}_3(y) - 3 \mathrm{Li}_3(-y) + \frac{1}{3} \mathrm{Li}_3(-y^3) \Big\}$$

where

$$y = \frac{1 - \sqrt{z/(z-4)}}{1 + \sqrt{z/(z-4)}}$$

Some other structures found by differential equation method

$$V_a(n) = \sum_{j=1}^{n} \frac{1}{\binom{2j}{j}} \frac{1}{j^a}$$

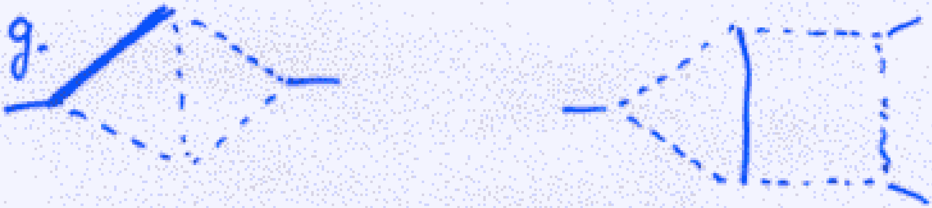
$$S_a(n;p) = \sum_{j=1}^{n} \frac{p^{j-n}}{j^a}$$

$$V_a(n;p) = \sum_{j=1}^{n} \frac{1}{\binom{2j}{j}} \cdot \frac{p^{j-n}}{j^a}$$

...

Threshold at $z = 1$

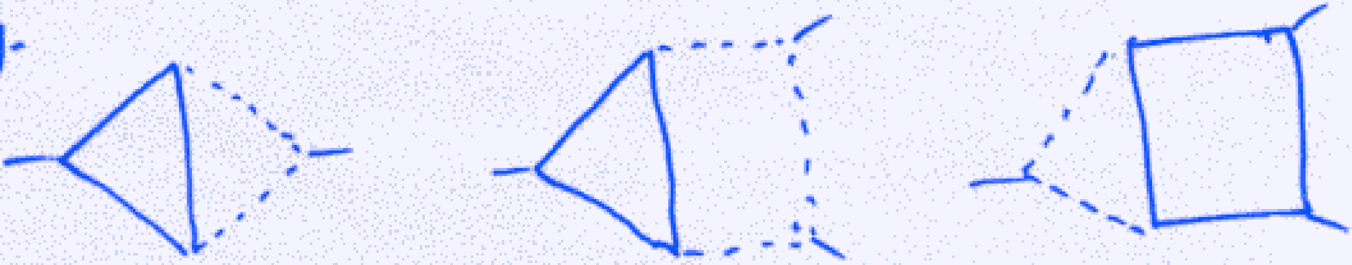
e.g.



ok.

Threshold at $z = 4 \cdot \frac{q^2}{m^2}$

e.g.



ok.

Mixed thresholds

e.g.

ok.

Threshold at $z = 9$

e.g.

?

e.g.

$$
{}_pF_q\left(\begin{matrix} 1+a_1\epsilon, \ldots, 1+a_r\epsilon \\ \frac{3}{2}+b\epsilon, 2+c_1\epsilon, \ldots, 2+c_s\epsilon \end{matrix} ; x\right)
$$

- How to expand in ϵ?
- What are the functions?

In case of $z_{th}=1$ – Remiddi-Vermaseren's functions (?)

In case $z_{th}>1$?

More simple proplem: consider single-scale diagrams. E.g. put $z=1$ (on-shell condition)

What are the constants?

In case of $z_{th}=1$: – multiple-zeta values

$$
\sum_{n_1 > n_2 \ldots > n_s} \frac{1}{n_1^{a_1} n_2^{a_2} \ldots n_s^{a_s}}
$$

In case of $z_{th}=4$: – binomial sums

$$
\sum_{n_1 n_2 \ldots} \frac{1}{\binom{2n}{n}} \frac{1}{n_1^{a_1} \ldots n_s^{a_s}}
$$

The basic elements are defined by the thresholds, but <u>not</u> by the number of loops or depth of ϵ-expansion.

$z_{th} = 1$: "even basis" $\qquad \omega_4 = \sqrt[4]{1}$

$z_{th} = 4$: "odd basis" $\qquad \omega_6 = \sqrt[6]{1}$

To construct basis take polylogarithms of arguments

Broadhurst

$$\omega, \omega^2, \omega^3, \ldots$$

$$1-\omega, 1-\omega^2, 1-\omega^3, \ldots$$

$$\frac{-\omega}{1-\omega}, \frac{-\omega^2}{1-\omega^2}, \frac{-\omega^3}{1-\omega^3}, \ldots$$

Basic elements form algebra, i.e.

if $\xi_1, \xi_2 \in B$ then

- $x_1 \xi_1 + x_2 \xi_2 \in B$
- $\xi_1 \xi_2 \in B$

Polylogarithms of complex argument.

L. Lewin, *Polylogarithms and associated functions* (North-Holland, Amsterdam, 1981).

$$\begin{aligned}
\mathrm{Li}_{2n}\left(e^{i\theta}\right) &= \mathrm{Gl}_{2n}(\theta) + i\mathrm{Cl}_{2n}(\theta)\,, \\
\mathrm{Li}_{2n+1}\left(e^{i\theta}\right) &= \mathrm{Cl}_{2n+1}(\theta) + i\mathrm{Gl}_{2n+1}(\theta)\,,
\end{aligned}$$

$$\begin{aligned}
\mathrm{Gl}_n(2\pi x) &= (-1)^{1+\left[\frac{n}{2}\right]} 2^{n-1}\pi^n \frac{B_n(x)}{n!}, \\
\frac{te^{xt}}{e^t-1} &= \sum_0^\infty B_n(x)\frac{t^n}{n!}. \\
\mathrm{Gl}_2(\theta) &= \frac{(\pi-\theta)^2}{12} - \frac{\pi^2}{12}, \\
\mathrm{Gl}_3(\theta) &= \frac{\theta}{12}(\pi-\theta)(2\pi-\theta)\,, \\
\mathrm{Gl}_4(\theta) &= \frac{\pi^4}{90} - \frac{\theta^2}{12}\left(\pi^2 - \pi\theta + \frac{\theta^2}{4}\right), \\
\mathrm{Gl}_5(\theta) &= \frac{\theta}{720}(\pi-\theta)(2\pi-\theta)\left(4\pi^2+6\pi\theta-3\theta^2\right).
\end{aligned}$$

$$\frac{1}{2^{n-1}}\mathrm{Cl}_n(2\theta) = \mathrm{Cl}_n(\theta) + \mathrm{Cl}_n(\pi+\theta)\,.$$

$$\begin{aligned}
\mathrm{Cl}_{2n+1}(2m\pi \pm \theta) &= \mathrm{Cl}_{2n+1}(\theta)\,, \\
\mathrm{Cl}_{2n+1}(\pm\theta) &= \mathrm{Cl}_{2n+1}(\theta)\,, \\
\frac{1}{2^{2n}}\mathrm{Cl}_{2n+1}(2\theta) &= \mathrm{Cl}_{2n+1}(\theta) + \mathrm{Cl}_{2n+1}(\pi-\theta)\,, \\
\mathrm{Cl}_{2n+1}\left(\frac{\pi}{3}\right) &= \frac{1}{2}\left(1-2^{-2n}\right)\left(1-3^{-2n}\right)\zeta_{2n+1}, \\
\mathrm{Cl}_{2n+1}\left(\frac{2\pi}{3}\right) &= -\frac{1}{2}\left(1-3^{-2n}\right)\zeta_{2n+1}.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Cl}_{2n}(2m\pi \pm \theta) &= \pm\mathrm{Cl}_{2n+1}(\theta)\,, \\
\mathrm{Cl}_{2n}(\pm\theta) &= \pm\mathrm{Cl}_{2n+1}(\theta)\,, \\
\frac{1}{2^{2n-1}}\mathrm{Cl}_{2n}(2\theta) &= \mathrm{Cl}_{2n}(\theta) - \mathrm{Cl}_{2n}(\pi-\theta)\,, \\
\mathrm{Cl}_{2n}\left(\frac{\pi}{3}\right) &= \left(1+2^{1-2n}\right)\mathrm{Cl}_{2n}\left(\frac{2\pi}{3}\right),
\end{aligned}$$

Polylogarithms of complex argument (continuation)

L. Lewin, *Polylogarithms and associated functions* (North-Holland, Amsterdam, 1981).

$$
\begin{aligned}
Re\mathrm{Li}_3\left(1-e^{i\theta}\right) &= \frac{1}{2}\zeta_3 - \frac{1}{2}\mathrm{Cl}_3(\theta) + \frac{\theta^2}{4}\ln\left(2\sin\frac{\theta}{2}\right), \\
Im\mathrm{Li}_3\left(1-e^{i\theta}\right) &= \frac{\theta^3}{24} - \frac{\theta}{2}\ln^2\left(2\sin\frac{\theta}{2}\right) - \mathrm{Cl}_2(\theta)\ln\left(2\sin\frac{\theta}{2}\right) + \frac{1}{2}\mathrm{Ls}_3(\theta), \\
Re\mathrm{Li}_4\left(1-e^{i\theta}\right) &= \frac{1}{4}\mathrm{Ls}_4^{(1)}(\theta) - \frac{\theta}{4}\mathrm{Ls}_3(\theta) + \frac{\theta^2}{8}\ln^2\left(2\sin\frac{\theta}{2}\right) \\
&+ \frac{1}{2}\left[\zeta_3 - \mathrm{Cl}_3(\theta)\right]\ln\left(2\sin\frac{\theta}{2}\right) - \frac{\theta^4}{192}, \\
Im\mathrm{Li}_4\left(1-e^{i\theta}\right) &= -\frac{1}{6}\mathrm{Ls}_4(\theta) + \frac{1}{2}\mathrm{Ls}_3(\theta)\ln\left(2\sin\frac{\theta}{2}\right) - \frac{\theta}{6}\ln^3\left(2\sin\frac{\theta}{2}\right) \\
&- \frac{1}{2}\mathrm{Cl}_2(\theta)\ln^2\left(2\sin\frac{\theta}{2}\right) + \frac{\theta^3}{24}\ln\left(2\sin\frac{\theta}{2}\right) - \frac{1}{4}\mathrm{Cl}_4(\theta) + \frac{\theta}{4}\zeta_3, \\
\mathrm{Li}_5\left(1-e^{i\pi/3}\right) &= \frac{25}{54}\zeta_5 - i\frac{85}{324}\pi\zeta_4, \\
\mathrm{Li}_5\left(1-e^{i2\pi/3}\right) &= -\frac{1}{12}\pi\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)\ln 3 + \frac{1}{18}\pi\mathrm{Ls}_4\left(\frac{2\pi}{3}\right) + \frac{1}{72}\zeta_2\ln^3 3 + \frac{13}{144}\zeta_3\ln^2 3 \\
&- \frac{5}{108}\zeta_4\ln 3 + \frac{121}{1296}\zeta_5 - \frac{1}{6}\zeta_2\zeta_3 + \frac{1}{8}\mathrm{Ls}_4^{(1)}\left(\frac{2\pi}{3}\right)\ln 3 - \frac{1}{12}\mathrm{Ls}_5^{(1)}\left(\frac{2\pi}{3}\right) \\
&+ i\left\{\frac{1}{576}\pi\ln^4 3 - \frac{1}{72}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^3 3 - \frac{1}{6}\zeta_2\mathrm{Ls}_3\left(\frac{2\pi}{3}\right) + \frac{1}{16}\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)\ln 3\right. \\
&- \frac{2}{81}\mathrm{Ls}_4\left(\frac{\pi}{3}\right)\ln 3 - \frac{1}{12}\mathrm{Ls}_4\left(\frac{2\pi}{3}\right)\ln 3 + \frac{1}{24}\mathrm{Ls}_5\left(\frac{2\pi}{3}\right) + \frac{1}{108}\pi\zeta_2\ln^2 3 \\
&+ \left.\frac{31}{324}\pi\zeta_3\ln 3 - \frac{1}{162}\pi\zeta_4 + \frac{1}{12}\pi\mathrm{Ls}_4^{(1)}\left(\frac{2\pi}{3}\right) - \frac{1}{16}\mathrm{Ls}_5^{(2)}\left(\frac{2\pi}{3}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Ls}_n(x) &= -\int_0^x \ln^{n-1}\left|2\sin\frac{\theta}{2}\right| d\theta, \\
\mathrm{Ls}_n^{(m)}(x) &= -\int_0^x \theta^m \ln^{n-m-1}\left|2\sin\frac{\theta}{2}\right| d\theta.
\end{aligned}
$$

PSLQ algorithm and results for multiple binomial sums

PSLQ: Find $r_1, r_2, \ldots, r_s$ such that

$$\xi_1 r_1 + \xi_2 r_2 + \ldots + \xi_s r_s = 0.$$

Borwein
Bailey

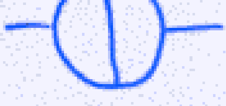
& where $r_1, r_2, \ldots$ are rational

e.g.

$$\sum_{n>0} \frac{1}{\binom{2n}{n}} \frac{1}{n^2} \sum_{j=1}^{n-1} \frac{1}{j} \sum_{k=1}^{n-1} \frac{1}{k} =$$

$$= \frac{4}{3} \pi \, Ls_3\left(\frac{2\pi}{3}\right) - \frac{16}{9} Ls_2^2\left(\frac{\pi}{3}\right) + \frac{1085}{108} \zeta_4$$

Two-loop self-energy diagrams on-shell



$$m^2(1-2\epsilon) I = r_1 \zeta_3 + r_2 \pi Ls_2\left(\frac{\pi}{3}\right) + r_3 i\pi \zeta_2$$

$$+ \varepsilon\left[r_4 \, Ls_2^2\left(\frac{\pi}{3}\right) + r_5 \pi \, Ls_3\left(\frac{2\pi}{3}\right) + r_6 \zeta_4 \right]$$

$$+ i\pi\varepsilon\left[r_7 \pi \, Ls_2\left(\frac{\pi}{3}\right) + r_8 \zeta_3 \right] + O(\varepsilon^2)$$

Dimensional Regularization:

F.D. schow algebraic structure (e.g. sixth root of unity algebra). Integration by part $\longleftrightarrow$ algebraic algorithms of reduction.

Lattic PT:

Is there any algebraic structure? If yes, are there algebraic algorithms of reduction

Structure : yes

Relation between basis and topology (?)

with D.S. Shiu (in progress)

- Level structure of the ϵ-expansion is studied.
  It is schown that the small number of basic function can be used to construct the basis of arbitrary weight
- Know of the basis allow to find analytical result
  - for off-shell case — solving linear equations
  - for on-shell case — PSLQ