1.10. Krummlinige Koordinatensysteme

bei einer speziellen geometr. Symmetrie (Kugel, Zylinder...) eines Problems sind kartes. Koordinaten oft kompliziert.

$\Rightarrow$ Transformation $(x, y, z) \to (u, v, w)$

mit eineindeutiger Abbildung:

$$x = x(u, v, w) \qquad u = u(x, y, z)$$

$$y = y(u, v, w) \qquad v = v(x, y, z)$$

$$z = z(u, v, w) \qquad w = w(x, y, z)$$

Koordinatenachsen $\Rightarrow$ Koordinatenlinien

= (Linien entlang derer die anderen Koordinaten konstant sind, z.B. Längengrade (θ und r konst.) und Breitengrade (φ und r const) bei Kugelkoordinaten)

Ortsvektor:

$$\vec{r} = (x, y, z)$$

$$\Rightarrow \quad \vec{r} = x(u,v,w)\,\vec{e}_x + y(u,v,w)\,\vec{e}_y + z(u,v,w)\,\vec{e}_z$$

$$d\vec{r} = \left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv + \frac{\partial x}{\partial w}dw\right)\vec{e}_x + \left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv + \frac{\partial y}{\partial w}dw\right)\vec{e}_y + (\ldots)\vec{e}_z$$

oder kurz

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u} du + \frac{\partial \vec{r}}{\partial v} dv + \frac{\partial \vec{r}}{\partial w} dw$$

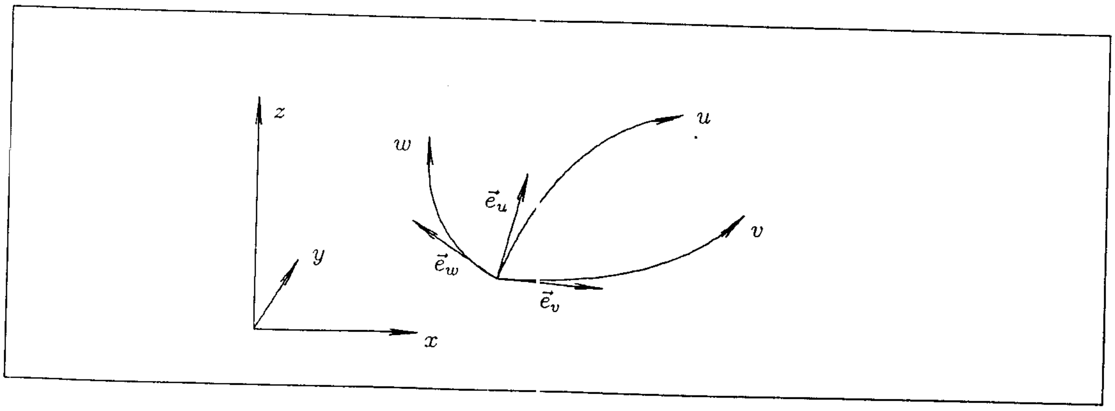
Metrische Koeffizienten:

$$\left|\frac{\partial \vec{r}}{\partial u}\right| \equiv g_u \quad \left|\frac{\partial \vec{r}}{\partial v}\right| \equiv g_v \quad \left|\frac{\partial \vec{r}}{\partial w}\right| \equiv g_w$$

heißen metr. Koeffizienten und beschreiben die Änderung des Maßstabs entlang u, v, w

Einheitsvektoren:

Die Ableitung $\frac{\partial \vec{r}}{\partial u}$ zeigt tangential entlang der Koordinatenlinie von u.



Damit sind die Einheitsvektoren in Tangentenrichtung:

$$\vec{e}_u = \frac{\partial \vec{r}}{\partial u} \Big/ \left|\frac{\partial \vec{r}}{\partial u}\right| \qquad \vec{e}_v = \frac{\partial \vec{r}}{\partial v} \Big/ \left|\frac{\partial \vec{r}}{\partial v}\right| \qquad \vec{e}_w = \frac{\partial \vec{r}}{\partial w} \Big/ \left|\frac{\partial \vec{r}}{\partial w}\right|$$

bzw. $\vec{e}_u = \frac{1}{g_u} \frac{\partial \vec{r}}{\partial u} \qquad \vec{e}_v = \frac{1}{g_v} \frac{\partial \vec{r}}{\partial v} \qquad \vec{e}_w = \frac{1}{g_w} \frac{\partial \vec{r}}{\partial w}$

Damit kann man $d\vec{r}$ in den neuen Koordinaten darstellen:

$$d\vec{r} = (d\vec{r} \cdot \vec{e}_u)\vec{e}_u + (d\vec{r} \cdot \vec{e}_v)\vec{e}_v + (d\vec{r} \cdot \vec{e}_w)\vec{e}_w$$

$\vec{e}_u, \vec{e}_v, \vec{e}_w$ bilden lokales Dreibein entlang der Koordinatenrichtungen.

$$\underline{d\vec{r} = g_u\, du\, \vec{e}_u + g_v\, dv\, \vec{e}_v + g_w\, dw\, \vec{e}_w}$$

Linienelement:

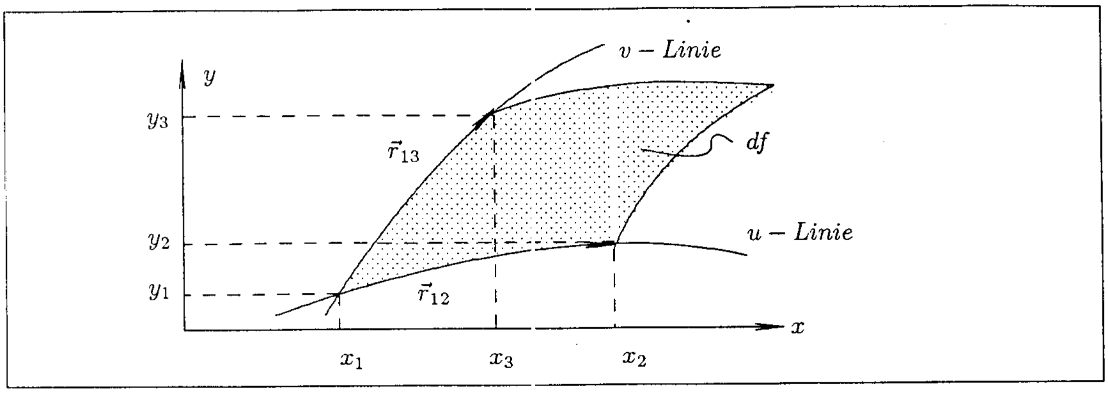
$$ds^2 = d\vec{r} \cdot d\vec{r} = \sum_{i=1}^{3} \sum_{j=1}^{3} g_i g_j\, du_i\, du_j\, (\vec{e}_i \cdot \vec{e}_j) =$$

$$\sum_{i=1}^{3} \sum_{j=1}^{3} g_{ij}\, du_i\, du_j \qquad (i,j = u,v,w)$$

Die 3x3 Matrix $g_{ij} = \frac{\partial \vec{r}}{\partial u_i} \frac{\partial \vec{r}}{\partial u_j} = g_i g_j (\vec{e}_i \cdot \vec{e}_j)$ heißt

Metrik des Koordinatensystems.

Flächenelement:



$\Delta\vec{f}$ sei ganz in der (x,y) und ganz in der (u,v)-Ebene.

<u>u-Linie</u>: $x_2 = x_1 + \frac{\partial x}{\partial u}\Delta u \quad y_2 = y_1 + \frac{\partial y}{\partial u}\Delta u$

<u>v-Linie</u>: $x_3 = x_1 + \frac{\partial x}{\partial v}\Delta v \quad y_3 = y_1 + \frac{\partial y}{\partial v}\Delta v$

(Taylor)

$$\Rightarrow \quad \vec{r}_{12} = \left(\frac{\partial x}{\partial u}\Delta u, \frac{\partial y}{\partial u}\Delta u, 0\right) = \frac{\partial \vec{r}}{\partial u}\Delta u$$

$$\vec{r}_{13} = \left(\frac{\partial x}{\partial v}\Delta v, \frac{\partial y}{\partial v}\Delta v, 0\right) = \frac{\partial \vec{r}}{\partial v}\Delta v$$

Wenn $\Delta\vec{f}$ klein ist, kann man es durch ein Parallelogramm nähern:

$$\Delta\vec{f} = \vec{r}_{12} \times \vec{r}_{13}$$

(Erinnerung: $\vec{a}\times\vec{b}$ steht senkrecht auf $\vec{a}$ und $\vec{b}$ hat den Betrag des Parallelogr.)

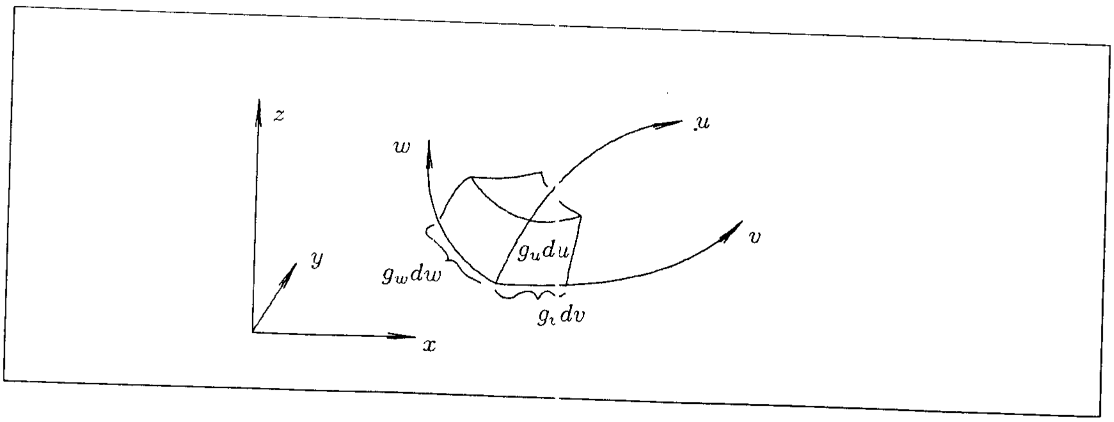
$$= \left(\frac{\partial \vec r}{\partial u} \times \frac{\partial \vec r}{\partial v}\right) \Delta u \, \Delta v$$

$$= \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix} \vec n \, \Delta u \, \Delta v$$

$$\equiv \frac{\partial(x,y)}{\partial(u,v)} \Delta u \, \Delta v \cdot \vec n$$

Jacobi-/Funktionaldeterminante

Volumenelement:



analoge Betrachtung

$$dV = dx\,dy\,dz = \frac{\partial(x,y,z)}{\partial(u,v,w)} \, du\,dv\,dw$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} \equiv \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Orthogonale Koordinaten:

Vereinfachung der Formeln, wenn

$$\vec{e}_i \cdot \vec{e}_j = \delta_{ij} \equiv \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Metrik:

$$g_{ij} = \begin{pmatrix} g_1^2 & 0 & 0 \\ 0 & g_2^2 & 0 \\ 0 & 0 & g_3^2 \end{pmatrix}$$

Damit wird:

- das Linienelement $\underline{ds^2 = g_u^2 du^2 + g_v^2 dv^2 + g_w^2 dw^2}$
- das Flächenelement $\underline{d\vec{f}} = \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) du\,dv$

$$= \left(g_u \vec{e}_u \times g_v \vec{e}_v \right) du\,dv$$

$$= \underline{g_u g_v \, du\,dv\, \vec{e}_w}$$

- das Volumenelement $\underline{dV = g_u g_v g_w \, du\,dv\,dw}$

- die Jacobi-Determinanten:

$$\frac{\partial(x,y)}{\partial(u,v)} = g_u g_v \quad \text{und} \quad \frac{\partial(x,y,z)}{\partial(u,v,w)} = g_u g_v g_w$$

Beispiel 1: Kugelkoordinaten:

$$x = r \sin\theta \cos\varphi$$
$$y = r \sin\theta \sin\varphi$$
$$z = r \cos\theta$$

Metrische Koeffizienten:

$$g_r = 1 \qquad g_\theta = r \qquad g_\varphi = r\sin\theta$$

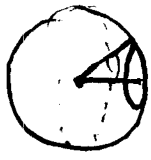
denn: $$g_r = \left| \frac{\partial \vec{r}}{\partial r} \right| = \left[\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 \right]^{1/2}$$

$$= \left[\sin^2\theta \cos^2\varphi + \sin^2\theta \sin^2\varphi + \cos^2\theta \right]^{1/2}$$
$$= \left[\sin^2\theta \left(\quad 1 \quad \right) + \cos^2\theta \right]^{1/2} = 1$$

$$g_\theta = \left[r^2\cos^2\theta\cos^2\varphi + r^2\cos^2\theta\sin^2\varphi + r^2\sin^2\theta \right]^{1/2} = r$$

$$g_\varphi = \left[r^2\sin^2\theta\sin^2\varphi + r^2\sin^2\theta\cos^2\varphi + 0 \right]^{1/2} = r\sin\theta$$

Linienelement: $$ds^2 = g_r^2 dr^2 + g_\theta^2 d\theta^2 + g_\varphi^2 d\varphi^2$$
$$= dr^2 + r^2 d\theta + r^2\sin^2\theta\, d\varphi$$

z.B.  Kreisumfang bei r, θ_0

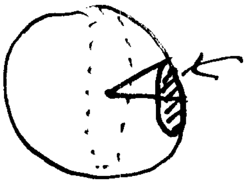
$$\int_0^{2\pi} ds = \int_0^{2\pi} r\sin\theta_0\, d\varphi = 2\pi r \sin\theta_0$$

Flächenelement: $d\vec{S} = g_\theta \, g_\varphi \, d\theta \, d\varphi \, \vec{e}_r$

(in θ, φ-Ebene)

$$= \underbrace{r^2 \sin\theta \, d\theta \, d\varphi}_{\text{"}d\Omega\text{" Raumwinkel}}$$

z.B. Kugelflächenstück



$$F = \int_0^{r} \int_0^{\theta_0} \int_0^{2\pi} r^2 \, dr \, \sin\theta \, d\theta \, d\varphi = 2\pi r^2 (1 - \cos\theta_0)$$

$\Rightarrow$ gesamte Raumwinkel für $r=1$: $\boxed{\int d\Omega = 4\pi}$

Volumenelement: $dV = g_r \, g_\theta \, g_\varphi \, dr \, d\theta \, d\varphi$

$\Rightarrow$ Kugelvolumen $\int dV = \int_0^{r} \int_0^{\pi} \int_0^{2\pi} r^2 \sin\theta \, dr \, d\theta \, d\varphi$

$$= \frac{4}{3}\pi r^3 \quad \checkmark$$

Beispiel 2: Zylinderkoordinaten

$$x = \rho \cos\varphi$$
$$y = \rho \sin\varphi$$
$$z = z$$

$$g_\rho = 1 \qquad g_\varphi = \rho \qquad g_z = 1$$

Linienelement: $ds^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2$

Flächenelement: $dF = \rho \, d\varphi \, dz$
(ρ = const.)

Volumenelement: $dV = \rho \, d\rho \, d\varphi \, dz$

Differentialoperatoren in orthog. krummlin. Koord.

1. Gradient:

$$d\phi = (\mathrm{grad}\,\phi)\, d\vec{r} \qquad \left(\begin{matrix}\text{allg.}\\ \text{Definition}\end{matrix}\right)$$

also: $$d\phi = (\mathrm{grad}\,\phi)\cdot\vec{e}_u\, g_u\, du + (\mathrm{grad}\,\phi)\cdot\vec{e}_v\, g_v\, dv + (\mathrm{grad}\,\phi)\vec{e}_w\, g_w\, dw$$

$$= (\mathrm{grad}\,\phi)_u\, g_u\, du + (\mathrm{grad}\,\phi)_v\, g_v\, dv + (\mathrm{grad}\,\phi)_w\, dw$$

$d\phi$ ist das totale Differential:

$$d\phi = \frac{\partial\phi}{\partial u}\, du + \frac{\partial\phi}{\partial v}\, dv + \frac{\partial\phi}{\partial w}\, dw$$

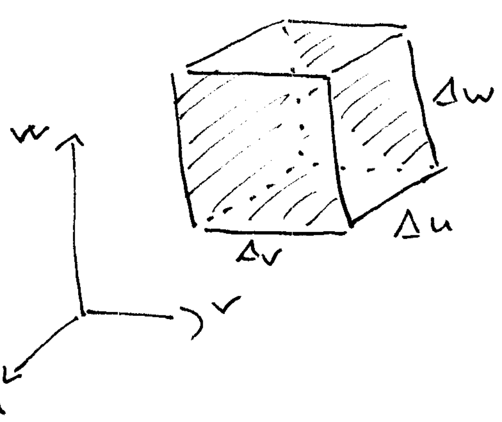
$$\Rightarrow \boxed{(\mathrm{grad}\,\phi)_u = \frac{1}{g_u}\frac{\partial\phi}{\partial u} \qquad (\mathrm{grad}\,\phi)_v = \frac{1}{g_v}\frac{\partial\phi}{\partial v} \qquad (\mathrm{grad}\,\phi)_w = \frac{1}{g_w}\frac{\partial\phi}{\partial w}}$$

2. Divergenz:

$$\mathrm{div}\,\vec{A} = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\oiint_{\Delta S}\vec{A}\cdot d\vec{S}$$

a) Fluß: Betrachte Volumen ΔV als Quader im (u,v,w)-System:

$$\Delta V = g_u g_v g_w\, \Delta u\, \Delta v\, \Delta w$$



Fluß durch die Flächen ist: $A_u(u+\Delta u)\Delta S(u+\Delta u) - A_u(u)\Delta S(u)$

mit $\Delta S = g_v g_w \Delta v \Delta w$

$\Rightarrow$ Fluß: $g_v(u+\Delta u)\, g_w(u+\Delta u)\, A_u(u+\Delta u)\, \Delta v\, \Delta w \; -$

$g_v(u)\, g_w(u)\, A_u(u)\, \Delta v \Delta w$

$$= \frac{\partial(g_v g_w A_u)}{\partial u} \cdot \Delta u \Delta v \Delta w$$

Gesamtfluß durch alle Flächen:

$$\oint\!\!\oint_{\Delta S} \vec{A} \cdot d\vec{S} = \left\{ \frac{\partial}{\partial u}(g_v g_w A_u) + \frac{\partial}{\partial v}(g_u g_w A_v) + \frac{\partial}{\partial w}(g_u g_v A_w) \right\} \Delta u \Delta v \Delta w$$

dieser Ausdruck muß gleich

$\operatorname{div} \vec{A} \cdot \Delta V = \operatorname{div} \vec{A}\, g_u g_v g_w \Delta u \Delta v \Delta w$ sein

$$\Rightarrow \boxed{\operatorname{div} \vec{A} = \frac{1}{g_u g_v g_w} \left\{ \frac{\partial}{\partial u}(g_v g_w A_u) + \frac{\partial}{\partial v}(g_u g_w A_v) + \frac{\partial}{\partial w}(g_u g_v A_w) \right\}}$$

3. Rotation: aus der allg. Definition der Rotation

$$\underbrace{(\operatorname{rot} \vec{A})_{\vec{n}}}_{\text{Komponente von } \vec{A} \text{ in } \vec{n}\text{-Richtung}} = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_{\Delta C} \vec{A} \cdot d\vec{r}$$

folgt analog:

$$\boxed{\operatorname{rot}\vec{A} = \frac{1}{g_u g_v g_w}\begin{vmatrix} g_u\vec{e}_u & g_v\vec{e}_v & g_w\vec{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ g_u A_u & g_v A_v & g_w A_w \end{vmatrix}}$$

(ohne Herleitung...)

4. Laplace-Operator:

Kombination der Formeln für grad und div:

$$\Rightarrow \boxed{\Delta\phi = \operatorname{div}\operatorname{grad}\phi = \frac{1}{g_u g_v g_w}\left\{\frac{\partial}{\partial u}\left(\frac{g_v g_w}{g_u}\right)\frac{\partial\phi}{\partial u} + \frac{\partial}{\partial v}\left(\frac{g_w g_u}{g_v}\right)\frac{\partial\phi}{\partial v} + \frac{\partial}{\partial w}\left(\frac{g_u g_v}{g_w}\right)\frac{\partial\phi}{\partial w}\right]}$$

Beispiel 1: Polarkoordinaten:

Gradient: $(\operatorname{grad}\phi)_r = \frac{\partial\phi}{\partial r}$ $\quad (\operatorname{grad}\phi)_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta}$ $\quad (\operatorname{grad}\phi)_\varphi = \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}$

Divergenz: $$\operatorname{div}\vec{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$

$$(\text{rot}\,\vec{A})_r = \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\varphi)}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\varphi}$$

$$(\text{rot}\,\vec{A})_\theta = \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\frac{\partial(rA_\varphi)}{\partial r}$$

$$(\text{rot}\,\vec{A})_\varphi = \frac{1}{r}\frac{\partial(rA_\theta)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial\theta}$$

Laplace-Operator:

$$\Delta\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$$

Beispiel 2 Zylinderkoordinaten:

Gradient: $(\text{grad}\,\phi)_\varrho = \frac{\partial\phi}{\partial\varrho}$ $(\text{grad}\,\phi)_\varphi = \frac{1}{\varrho}\frac{\partial\phi}{\partial\varphi}$ $(\text{grad}\,\phi)_z = \frac{\partial\phi}{\partial z}$

Divergenz: $\text{div}\,\vec{A} = \frac{1}{\varrho}\frac{\partial(\varrho A_\varrho)}{\partial\varrho} + \frac{1}{\varrho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}$

Rotation: $(\text{rot}\,\vec{A})_\varrho = \frac{1}{\varrho}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}$ $(\text{rot}\,\vec{A})_\varphi = \frac{\partial A_\varrho}{\partial z} - \frac{\partial A_z}{\partial\varrho}$

$$(\text{rot}\,\vec{A})_z = \frac{1}{\varrho}\frac{\partial(\varrho A_\varphi)}{\partial\varrho} - \frac{1}{\varrho}\frac{\partial A_\varrho}{\partial\varphi}$$

Laplace: $\Delta\phi = \frac{1}{\varrho}\frac{\partial}{\partial\varrho}\left(\varrho\frac{\partial\phi}{\partial\varrho}\right) + \frac{1}{\varrho^2}\left(\frac{\partial^2\phi}{\partial\varphi^2}\right) + \frac{\partial^2\phi}{\partial z^2}$