Introduction to String Theory exercises - sheet 5

Exercise 1

In this exercise it is shown that the Veneziano amplitude,

$$A_4(s,t) = \frac{\Gamma(-\alpha' s - \alpha_0)\Gamma(-\alpha' t - \alpha_0)}{\Gamma(-\alpha'(t+s) - 2\alpha_0)}$$

can only be interpreted as part of a unitary quantum field theory if the number of dimensions obeys $D \leq 26$ and the 'intercept' obeys $\alpha_0 = 1$[1].

a. What is the mass dimension of α'? What sets its natural size?

b. Locate the poles in the Veneziano amplitude in the $s = -(k_1 + k_2)^2$ and $t = -(k_2 + k_3)^2$ channel. (Bonus: locate the zeros as well.)

c. Show using either unitarity or a Feynman graph based argument

$$\lim_{\alpha' s \to X} (\alpha' s - X) A_4(s,t) = \sum_{\text{spectrum}} A_3^L A_3^R$$

should hold, where X is the location of one of the poles found in b. Moreover, from the explicit expression above,

$$\lim_{\alpha' s \to X} (\alpha' s - X) A_4(s,t) = \frac{1}{n!} \prod_{i=0}^{n} \left(\alpha' t + \alpha_0 + i\right)$$

with n related to X.

d. The external particles on the Veneziano amplitude must be scalars, of mass m. Show that the three point amplitudes in c must be of the form

$$A_3^R(T, T, M^{(j)}) \propto \prod_i^j \sqrt{\frac{\alpha'}{2}} (k_1 - k_2)^{\mu_i} \xi^{I_i}_{\mu_i}$$

for some integer j. Here ξ^I_μ is the polarization vector of a massive or massless vector boson, depending on the particle M.

e. Argue that j would be the (maximal) spin in $D = 4$.

f. Show that the right hand side in c must be

$$A_3^L A_3^R = \left(\alpha' t - 2\alpha' m^2 + \frac{X}{2}\right)^j$$

(Hint: start with $j = 1$)

[1]See also P. Frampton, Phys.Lett. B41

g. Show that the bound $j \leq n$ holds by inspecting the order of the polynomials at the residue.

h. Interpreting the lowest mass pole of the amplitude as the same particle type as those on the 'outside', show this fixes $m^2\alpha' = -1$.

i. From the next-to-lowest mass pole, obtain $\alpha_0 = 1$ by demanding that the three point coupling constant is real, and restricting to only odd-spin particles at odd levels

j. From the next-to-next-to lowest mass pole, obtain $D \leq 26$ by factorising contributions into *irreps* of SO(D-2) (hint: the dimension is in a tracelesness condition for a symmetric tensor)

k. Bonus 1: based on the fact that the external particles are bosons, argue that the amplitude as given above is in this form incomplete. Obtain the most generic completion you can think of.

l. Bonus 2: show the SO(D-2) Dynkin labels of the states coupling in amplitudes of the type obtained in d are $(\lambda, 0, 0, \ldots)$. Compare to equation 3.27 in 1007.2622 [hep-th]. Do all states couple to two tachyons?

This result on D and α_0 (ultimately proven through other means) is known as the 'no ghost' theorem in bosonic string theory.