# QFT II exercises - sheet 6

If you find a mistake in this exercise first check the website if the problem has been resolved already in a newer version. Otherwise, please email rutger.boels@desy.de.

**Exercise 1**

Consider the Lagrangian of Yang-Mills theory

$$\mathcal{L} = \sum_j \left(\overline{\psi}_j i\gamma^\mu (D_\mu\psi)_j - m\overline{\psi}_j\psi_j\right) - \frac{1}{4c(R)}\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right)$$

with

$$(D_\mu\psi)_j = \partial_\mu\psi_j - ig\sum_k (A_\mu)_{jk}\psi_k \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$$

a Show

$$[D_\mu, D_\nu]\psi = -igF_{\mu\nu}\psi$$

b By using $(A_\mu)_{ij} = \sum_a A^a_\mu t^a_{ij}$ show

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - ig\sum_{b,c} f^{abc}A^b_\mu A^c_\nu$$

c Show that

$$D_\rho F_{\mu\nu} \equiv \partial_\rho F_{\mu\nu} - ig[A_\rho, F_{\mu\nu}]$$

defines a covariant derivative. Determine $D_\rho F^a_{\mu\nu}$

d Show that the Euler-Lagrange equations derived from the action above are given by

$$i\gamma^\mu (D_\mu\psi)_j - m\psi_j = 0 \qquad D^\mu F^a_{\mu\nu} = gj^a_\nu$$

and compute $j^a_\nu$.

e Compute the Noether current for the global gauge transformation and demonstrate it is conserved using the equations of motion.

f Show that

$$D_\rho F_{\mu\nu} + D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} = 0$$

holds and that this equation is equivalent to $D_\rho(\epsilon_{\sigma\rho\mu\nu}F^{\mu\nu}) = 0$

g Argue that any gauge field for which

$$F_{\sigma\rho} \propto \epsilon_{\sigma\rho\mu\nu}F^{\mu\nu}$$

holds is a solution to the classical equations of motion. (aside:a special case belonging to this class is known as an 'instanton'. These have various uses in physics and mathematics)

**Exercise 2**

a Show that the matrices $(t^a)^{bc} \equiv i f^{abc}$ are a representation of any Lie algebra, i.e., that they satisfy

$$[t^a, t^b] = i f^{abc} t^c$$

b Define $T^2_{ij} = \sum_a t^a t^a$ for any representation matrices $t$. Show this operator commutes with any representation matrix,

$$[T^2, t^b] = 0 \qquad \forall b$$

c Use $T^2 = c_2(r)\mathcal{I}$ and $\mathrm{tr} t^a t^b = c(r)\delta^{ab}$ to show

$$c(r)d(G) = c_2(r)d(r)$$

d Check this equation for the fundamental representation of $SU(2)$. Hint: you should encounter Pauli-matrices.

**Exercise 3**

In the lecture it was shown briefly that the Jacobi identity can be represented graphically as the 'IHX relation'

a Introduce 'Feynman rules': a 'propagator' and a 'three vertex' and write the Jacobi relation as a sum of three Feynman graphs with two vertices and four external legs.

b Pick any two outside legs to be special. Show that any connected tree-level Feynman graph with $n$ external legs can be expressed as a sum over all tree level graphs with the maximal distance between the two special outside legs. Argue that this basis has dimension $(n-2)!$. Hint: this is best solved by drawing pictures! Fact: the number of all tree level graphs in this case is $(2n-5)!!$.

c Similarly, show that any one loop graph can be expressed in terms of those graphs which form a ring, with all outside legs attaching directly to the ring (no tree level outgrowth). Argue that this basis has dimension $\propto (n-1)!$.

d BONUS: find the similar number for two loop graphs as a general function of the number of legs. WARNING: this is a research level question, whose outcome may be publishable; the answer for four, five and six legs is known to be (bounded by) 5, 22 and 120 respectively. This, unfortunately, is not in the online integer sequence encyclopaedia (www.oeis.org), which is a standard method for guessing the solution to / solving counting problems.