QFT II exercises - sheet 2

v1.1: includes some bug fixes...

Exercise 1

a Show

$$\int (\prod_m d\theta_m d\theta_m^*) e^{\sum_{ij} \theta_i^* B^{ij} \theta_j} = \det B$$

where θ_m are Grassmann variables and B_{ij} a non-Grassmann-valued Hermitean matrix. Hint: first do the computation for one θ. For more variables, think of a similar problem with bosonic integration.

b Show

$$\int (\prod_m \theta_m d\theta_m^*) \theta_l^* \theta_k e^{\sum_{ij} \theta_i^* B^{ij} \theta_j} = (\det B) B_{kl}^{-1}$$

Exercise 2

The fermion propagator is given by

$$S(x-y) = \int \frac{dp^4}{(2\pi)^4} \frac{\not{p} + m}{p^2 - m^2} e^{-\mathrm{i}p(x-y)}$$

and is a Green's function for the Dirac operator, i.e.

$$(\mathrm{i}\gamma^\mu \partial_\mu - m) S(x-y) = i\delta^4(x-y)$$

a Show

(a) $\gamma^0 S^\dagger(x-y)\gamma^0 = -S(y-x)$

(b) $(\mathrm{i}\partial_x^\mu S(y-x)\gamma_\mu + mS(y-x) = i\delta^4(x-y)$

b Show that the generating functional for the free Dirac theory reads

$$Z[\bar{\eta}, \eta] := \int D\psi D\bar{\psi} e^{\mathrm{i}\int d^4x(\mathcal{L}_0 + \bar{\psi}\eta + \bar{\eta}\psi)} = Z[0] e^{-\int d^4x d^4y [\bar{\eta}(x) S(x-y) \eta(y)]}$$

Remember, $\mathcal{L}_0 = \bar{\psi}(i\not{\partial} - m)\psi$. For bonus points, argue in three lines (not more!) that this has to be the right form.

c Compute

$$\langle 0 | T\{\psi(x_1)\psi(x_2)\bar{\psi}(x_3)\bar{\psi}(x_4)\} | 0 \rangle$$

by taking appropriate derivates of $Z[\bar{\eta}, \eta]$

Exercise 3

In this exercise we will derive the propagator for the photon field A_μ in electrodynamics, $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ in light cone gauge:

$$q_\mu A^\mu = 0 \qquad q^2 = 0$$

Use the following steps:

1. First briefly argue that the answer should read something like

$$\langle 0|T\{A_\mu A_\nu\}|0\rangle \propto \int d^4k e^{-k\cdot(x-y)} \frac{1}{k^2}\left(\eta_{\mu\nu} + \alpha\frac{(q_\mu k_\nu + q_\nu k_\mu)}{q\cdot k}\right)$$

   based on Poincare symmetry, translation invariance, and symmetry in the indices. α is a constant which can be fixed by contracting this Ansatz with q.

2. Consider the path integral for the generating functional (i.e. including a source J_μ). Derive its field equation.

3. Solve this equation using the gauge condition

4. Using the solution, shift the field in the path integral to 'complete the square'

5. Write the answer as some infinite constant times an exponential with the source field appearing quadratically in the exponent

6. Derive the propagator $\langle 0|T\{A_\mu A_\nu\}|0\rangle$ by a double functional derivative.

Exercise 4: Easter BONUS

Consider the quantum field theory defined by

$$\mathcal{L} = \frac{1}{2}\left(\frac{1}{2}B_{\mu\nu}B^{\mu\nu} - B_{\mu\nu}F^{\mu\nu}\right)$$

where F is the electromagnetic field strength tensor. This theory has two different fields: A_μ and $B_{\mu\nu}$. Perform the path integral over B to show this theory is equivalent to electrodynamics. Derive the propagators for the B and A fields in the light cone gauge. The photon propagator should look familiar. (this computation is in the literature in Phys.Lett. B401 (1997) 62-68. If you put "hep-th/9702035" into google, you should be able to find this article)