QFT II exercises - sheet 10

If you find a mistake in this exercise first check the website if the problem has been resolved already in a newer version. Otherwise, please email rutger.boels@desy.de.

Exercise 1

Consider an $SU(2)$ gauge theory with a scalar in the real vector representation. It's Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu,a} F^a_{\mu\nu} + \frac{1}{2} (D_\mu \phi)^i (D^\mu \phi)^i - V\left(\phi^i\right)$$

with

$$V = -\frac{1}{2}\mu^2 \phi^i \phi^i + \frac{1}{4}\lambda(\phi^i\phi^i)^2 \quad (D_\mu \phi)^i = \partial_\mu \phi^i + g\epsilon^{ijk} A^j_\mu \phi^k$$

and $\mu^2 > 0$ as well as $\lambda > 0$

a Minimize the potential to find the vacuum. Write the fluctuations about the vacuum in terms of fields ϕ^1, ϕ^2 and $\phi^3 = v + h(x)$

b identify the would-be Goldstone bosons and compute the mass of the Higgs boson.

c Identify the broken and unbroken generators of the gauge transformations by studying the transformations of the vacuum. Determine the unbroken gauge group.

d How do the massive gauge bosons transform under an unbroken gauge symmetry transformation?

e Translate the previous answer into the charge of the gauge bosons

Exercise 2

The electroweak sector of the standard model contains three massive and one massless gauge boson. Two of these have an electrical charge.

a For a gauge group $U(1)^4$ find an assignment of charges for the Higgs field and a potential for that field such that one obtains 3 massive and one massless gauge boson.

b Under what condition are two of these gauge boson masses equal?

c Show the resulting massive gauge bosons are always uncharged under the residual gauge group (i.e. the U(1) of would-be electrodynamics). This rules out $U(1)^4$ as a physical theory d For the gauge group $SU(2) \times U(1)$ and the Higgs boson in the spinor representation of $SU(2)$ but uncharged under the $U(1)$, show one obtains three equally massive, uncharged gauge bosons, as well as one massless one.

e For the gauge group $SU(2) \times U(1)$ and the Higgs boson charged under $U(1)$ but uncharged under the $SU(2)$, show one obtains three massless uncharged gauge bosons, as well as one massive one.

f For the gauge group $SU(2) \times U(1)$ and the Higgs boson in the spinor representation of $SU(2)$ and charged under the $U(1)$, work out the masses and charges of the resulting gauge bosons.

Exercise 3

Consider an unbroken gauge theory in six dimensions given by the usual Yang-Mills Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu,a}$$

where now the indices run from 0 to 5.

a Split the indices into two sets: indices $a, b, \ldots$ running from 0 to 3 and indices $I, J, \ldots$ running from $4, 5$. Derive the Lagrangian obtained by setting all derivatives in the 4,5 direction to zero. This is sometimes called 'dimensional reduction'

b Show that the resulting Lagrangian is a Yang-Mills theory in four dimensions, coupled to a complex scalar $\phi = \frac{1}{\sqrt{2}}(A_4 + \mathrm{i}A_5)$ that transforms in the adjoint.

c Show that the difference of the resulting theory to the minimally coupled one is a potential term $\sim [\phi, \bar{\phi}]^2$

d Show that the potential is minimised by $\phi = \sum_a c_a H^a$, for some vector of complex constants c_a where H^a are the generators in the adjoint that commute with all the adjoint generators of the Lie algebra (i.e. H spans the Cartan subalgebra)

e Argue this vev breaks (some) of the gauge symmetry. Argue that there are at least as many massless vector bosons as elements H^a

f BONUS: argue that the effect of the vev can be reinterpreted as giving certain fields a momentum in the directions 4 and 5. Hint: study the effect of the covariant derivative on the gauge field. You can assume that the generators that do not commute with H^a can be grouped into eigenvectors of H^a which can appear with either sign:

$$[H^a, e_k^\pm] \propto \pm \lambda_k^a e_k^\pm \qquad \text{no sum on } k$$