QFT II exercises - sheet 0

Exercise 1

a Show, assuming $a \neq a(x)$ and $J \neq J(x)$ that

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2+Jx} = \sqrt{\frac{2\pi}{a}} e^{\frac{J^2}{2a}}$$

b Show, assuming $A \neq A(x)$ and $J \neq J(x)$ and with summation convention as well as a suitable constraint on the matrix A that Z[J] :=

$$\int_{-\infty}^{\infty} dx_1 \dots dx_n e^{-\frac{1}{2}(x_i A_{ij} x_j)+J_i x_i} = \sqrt{\frac{(2\pi)^n}{\det(A)}} e^{\frac{1}{2}J_i J_j (A^{-1})_{ij}}$$

Hint: Diagonalize A.

c Compute

$$\frac{1}{Z[0]} \frac{\partial^4 Z[J]}{\partial J_i \partial J_j \partial J_k \partial J_l} \lfloor_{J=0} = \int_{-\infty}^{\infty} dx_1 \dots dx_n \, x_i \, x_j \, x_k \, x_l \, e^{-\frac{1}{2}(x^T A x)}$$

d Formulate a simple ‘Feynman rule’ which reproduce the result of c).

Exercise 2

A one dimensional harmonic oscillator with external force J(t) is described by the Lagrangian

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega x^2 + Jx$$

a Give the action in the form

$$S = \int dt \left[\frac{1}{2} x A x + Jx\right]$$

and determine the differential operator A, assuming surface terms vanish

b The Green’s function $G(t-t')$ of this operator satisfies

$$A(t)G(t-t') = \delta(t-t')$$

Show

$$S = \int dt[\frac{1}{2}x' A x' - \frac{1}{2}\int dt' J(t)G(t-t')J(t')]$$

for a suitably shifted $x' = x + X$. *Hint:* X must be linear in J and involve an integral

c Give the Fourier representation of $G(t-t')$

Exercise 3

The action for a gauge boson A_μ is given by

$$S = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

a By using partial integration write the action in the form

$$S = \frac{1}{2}\int d^4x A_\mu D^{\mu\nu} A_\nu$$

and determine the differential operator $D^{\mu\nu}$

b By using the Ansatz $A_\mu(x) = \int \frac{d^4k}{(2\pi)^4}\tilde{A}_\mu e^{-\mathrm{i}k\cdot x}$ show that the action if Fourier space takes the form

$$S = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\tilde{A}_\mu(k)\tilde{D}^{\mu\nu}\tilde{A}_\nu(-k)$$

and compute $\tilde{D}^{\mu\nu}(k)$.

c The Greens function $G_{\nu\rho}(x-y)$ of $D^{\mu\nu}$ is defined by

$$D^{\mu\nu}G_{\nu\rho}(x-y) = i\delta^\mu{}_\rho\delta(x-y) \tag{1}$$

Show that $G_{\nu\rho}(x-y)$ is ill defined by acting with ∂_μ on equation (1)

d By using the Ansatz $G_{\nu\rho}(x-y) = \int \frac{d^4k}{(2\pi)^4}\tilde{G}_{\nu\rho}e^{\mathrm{i}k(x-y)}$ determine the analog of equation (1) for $\tilde{G}_{\nu\rho}$. How does the problem of c) show up in this equation?

e Add to the action S a term

$$\delta S = -\frac{1}{2\xi}\int d^4x(\partial_\mu A^\mu)^2$$

and recompute $D^{\mu\nu}$ and $\tilde{D}^{\mu\nu}$. Is the problem of c) still there?

f Determine $\tilde{G}_{\nu\rho}$ with the help of the Ansatz

$$\tilde{G}_{\nu\rho} = a(k^2)\eta_{\nu\rho} + b(k^2)k_\nu k_\rho$$

and compute a and b.

g BONUS: instead of e), take

$$\delta S = -\frac{1}{2}\int d^4x(q_\mu A^\mu)^2$$

for some light-like vector q ($q^2 = 0$). Determine $\tilde{G}_{\nu\rho}$ with the help of the Ansatz

$$\tilde{G}_{\nu\rho} = a(k^2)\eta_{\nu\rho} + b(k^2)\left(q_\nu k_\rho + q_\rho k_\nu\right)$$