Scattering Amplitudes, Integrability, and the Multi-Regge Limit

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Classical and quantum symmetries in mathematics and physics,
Jena, July 25, 2016

TB, G. Papathanasiou, V. Schomerus: The 2-Loop Symbol of all Multi-Regge Regions
TB: Systematics of the Multi-Regge Three-Loop Symbol
TB, Vsevolod Chestnov, Volker Schomerus: work in progress

Motivation: Why Amplitudes?

Amplitudes are interesting for many reasons!

- Actual link between QFT and particle physics
- Interesting as mathematical objects (special functions)
- Treasure box of interesting properties and relations
- Probe for properties of underlying theories

Another angle:

- Want to understand QFT at weak, strong, and at finite coupling
- Now: locality, gauge invariance, Lagrangean
  $\rightarrow$ weak coupling, divergences, complicated (gauge redundancies)!
- Amplitudes (and other objects) enjoy massive simplifications
- $\rightarrow$ Find other principles? (unitarity, consistency, symmetries?, . . . ?)
- Amplitudes as a laboratory for properties!

The Simplest QFT

Start with the simplest QFT: $\mathcal{N} = 4$ super Yang–Mills theory

- Cousin of QCD (massless, adjoint matter, supersymmetry)
- Many interesting properties and simplifications
- Conformal symmetry (β-function vanishes)
- AdS/CFT duality to IIB strings (quantum gravity)
- Integrable in planar limit

Planar theory: Hydrogen atom of 21st century

Solve planar theory at finite coupling! [Beisert et al '10] [Gromov, Kazakov Leurent, Volin '13]

- Spectrum of anomalous dimensions essentially solved
- Now: Dynamics of the theory $\rightarrow$ scattering amplitudes!

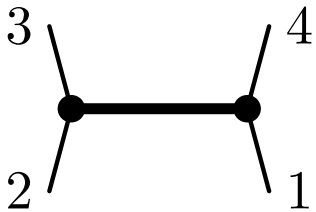
The Multi-Regge Limit

Amplitudes are complicated functions of many variables!
Simplify problem: Take limits.

Regge limit

- Large forward energies s_i, fixed momentum transfer & angles
- High-energy scattering $\rightarrow$ Collider computations
- Studied for many decades in QCD
- Expansion in large logarithms $\log(s_i)$
- Characterized by resummation and exponentiation
- Effective dynamical states governed by integrable Hamiltonian
- Reggeized gluon solves s-channel unitarity to all orders

$$\mathcal{A}_4^{\mathrm{MRL}} = \quad = \Gamma(t) s^{\omega(t)} \Gamma(t)$$



Amplitudes in the Simplest QFT

Witten's twistor string description [Witten 0312171]

Rapid progress due to unexpected properties and hidden symmetries

- New computational (on-shell!) methods
  (avoid all gauge redundancies, only use physical quantities)
- Dual conformal & Yangian symmetry [Drummond, Henn Korchemsky, Sokatchev] [Drummond Henn, Plefka]
- Computations to unprecedented loop orders and multiplicities

Finite-coupling, integrability-based description in multi-collinear limit:
Wilson loop OPE [Basso, Sever Vieira '13]

- Based on duality to null polygon Wilson loops
- Multi-collinear limit: WL flattened to square
- Sum over flux tube excitations over GKP vacuum
- OPE $\leftrightarrow$ Pentagon transitions
- Pentagon transitions factorize (integrability)

Multi-Regge Limit in $\mathcal{N} = 4$ SYM

Mission: Revisit multi-Regge limit in the light of new structures and results in $\mathcal{N} = 4$ SYM amplitudes.

Two-fold motivation:

Solve $\mathcal{N} = 4$ SYM amplitudes in MRL

- Different kinematic limit than multi-collinear limit
- Limit is known for special properties $\rightarrow$ solvable?! (resummation, exponentiation, BFKL integrability)
- More data for going to full kinematics

Understand MRL quantitatively

- Decomposition of amplitude in Regge cut terms
- Structure of Regge cut terms
- Universality of Regge physics: May apply to other theories
- Regge theory partly heuristic: Test it!

BDS Amplitude and Remainder Function

Scattering amplitude splits:

$$\mathcal{A}_n(g) = \mathcal{A}_n^{\mathrm{BDS}}(g) R_n(g)$$

Bern–Dixon–Smirnov (BDS) part: [Bern, Dixon Smirnov '05]

- Well-understood
- Exponential: $\mathcal{A}_n^{\mathrm{BDS}} = \mathcal{A}_n^{\mathrm{tree}} \exp(\mathcal{A}_n^{\mathrm{one-loop}})$
- Captures all IR & UV divergences and all dual conformal weights
- Correct amplitude for $n = 4, 5$ ($R_4 = R_5 = 1$)

Remainder function $R_n(g)$: [Bartels, Lipatov Sabio Vera '08] [Lipatov 1008.1015]

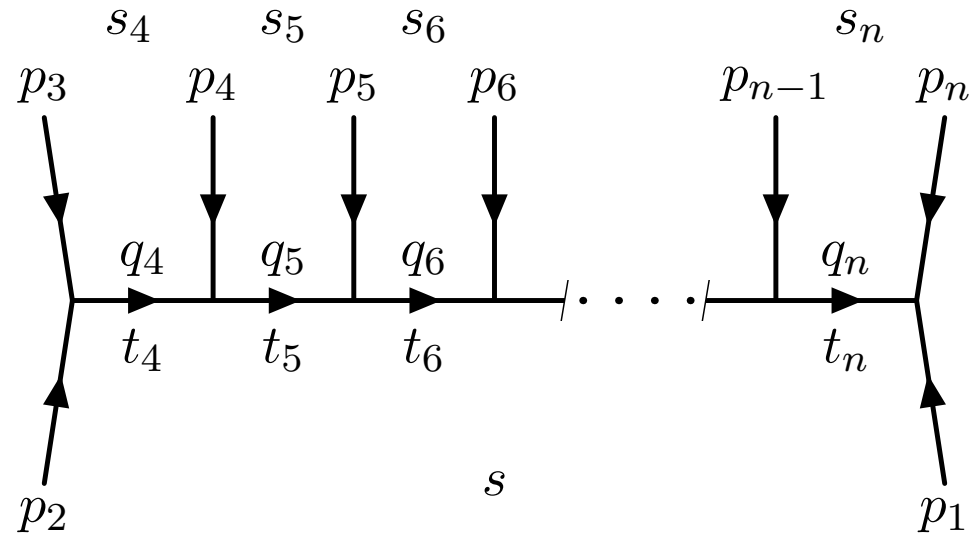
- Finite, dual conformally invariant
- Function of $(3n - 15)$ independent invariant cross ratios

$$U_{ij} = \frac{x_{i,j+1}^2 x_{i+1,j}^2}{x_{ij}^2 x_{i+1,j+1}^2}\,, \qquad x_i - x_{i-1} \equiv p_i\,.$$

- Starts at two loops and $n = 6$

Multi-Regge Limit and Kinematic Regions

Kinematics:



Multi-Regge limit (MRL):

$$|s| \gg |s_4|, \ldots, |s_n| \gg |t_4|, \ldots, |t_n|\,, \qquad s_j \equiv (p_{j-1} + p_j)^2\,, \quad t_j \equiv q_j^2\,.$$

In order to understand analytic structure of MRL (and full) amplitude, need to consider different **kinematic regions**:

Starting region: Particles $3, \ldots, n$ outgoing ($p_3^0, \ldots, p_n^0 > 0$, all $s_j < 0$)
Other ("Mandelstam") regions:

Some energies continued (particles become incoming, some $s_j > 0$)

Amplitudes pick up various terms upon continuation.

Functions for Amplitudes

Large classes of Feynman integrals (and thus amplitudes) can be expressed in terms of **iterated integrals** [Chen 1973]

$$I(X_1, \ldots, X_k) \equiv \int_{0 \leq t_1 \leq \cdots \leq t_k \leq 1} \mathrm{dlog}(X_1(t_1)) \ldots \mathrm{dlog}(X_k(t_k))$$

with X_k are rational functions in the space $\mathcal{X}$ of kinematic variables that depend on the parameters t_i through a common curve $\gamma : [0, 1] \to \mathcal{X}$.

For remainder function: $\mathcal{X}$ is the space of $(3n - 15)$ independent dual conformally invariant cross ratios U_{ij}

Sequence $\{X_1, \ldots, X_k\}$ satisfies **integrability condition**
$\Rightarrow I$ only depends on endpoints and homotopy of γ in $\mathcal{X} \setminus \mathcal{Z}$, where $\mathcal{Z}$ is the set of zeros and singularities of $X_1, \ldots, X_k$.
$\Rightarrow I$ is multivalued function on $\mathcal{X} \setminus \mathcal{Z}$.

Example: Multiple polylogs $\mathrm{Li}_{n_1,\ldots,n_k}$ [Goncharov 0103059]

From Functions to Symbols

Iterated integrals satisfy many functional identities
Expressions with many terms become unwieldy
Difficult/impossible to simplify

$\rightarrow$ Reduce iterated integrals to *symbols*

Symbol map: Linear map on space of iterated integrals [Goncharov 0908.2238]

$$\int_{0\leq t_1\leq\cdots\leq t_k\leq 1} \mathrm{dlog}(X_1(t_1))\ldots\mathrm{dlog}(X_k(t_k)) \mapsto (X_1\otimes\cdots\otimes X_k)$$

At the symbol level, functional identities become algebraic
Easy to write expressions in canonical form

Price: Symbol forgets information on path, determines original iterated integral only up to monodromies (functions of lower transcendental weight). Symbols of monodromies are easy:

$$S[\operatorname*{disc}_{\eta} I(X_1,\ldots,X_k)] = \left[\oint_{\eta}\mathrm{dlog}(X_1)\right](X_2\otimes\cdots\otimes X_k)$$

The Two-Loop Symbol

Recall: $\mathcal{A}_n(g) = \mathcal{A}_n^{\text{BDS}}(g) R_n(g)$

Remainder function R_n starts at $n = 6$ and two loops [Del Duca, Duhr, Smirnov '09]

Seven-point $R_7^{(2)}$ also known, but more complicated [Golden, Spradlin]

Six-point $R_6^{(2)}$ simplifies dramatically [Goncharov, Spradlin, Vergu, Volovich]

At symbol level: $R_n^{(2)}$ known at all n! [Caron-Huot 1105.5606]

Want to understand **discontinuities**. Properties:

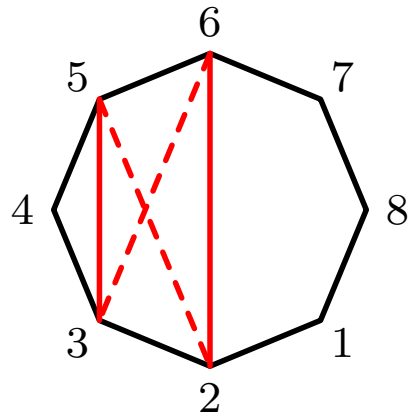
- Remainder function is sum of iterated integrals
- Branch cuts $\Leftrightarrow$ intermediate particle on-shell (by unitarity)
  $\Leftrightarrow$ in planar massless theories: $x_{ij}^2 = 0$, $x_i - x_{i-1} = p_i$
- Dual conformal invariance: R_n can only depend on cross ratios

$$U_{ij} = \frac{x_{i,j+1}^2 x_{i+1,j}^2}{x_{ij}^2 x_{i+1,j+1}^2}\,, \qquad x_i - x_{i-1} \equiv p_i\,.$$

- $\Rightarrow$ First symbol entries must be from set of U_{ij}.

Cross Ratios and Winding Numbers

Cross ratios: $U_{ij} = \frac{x^2_{i,j+1} x^2_{i+1,j}}{x^2_{ij} x^2_{i+1,j+1}}$



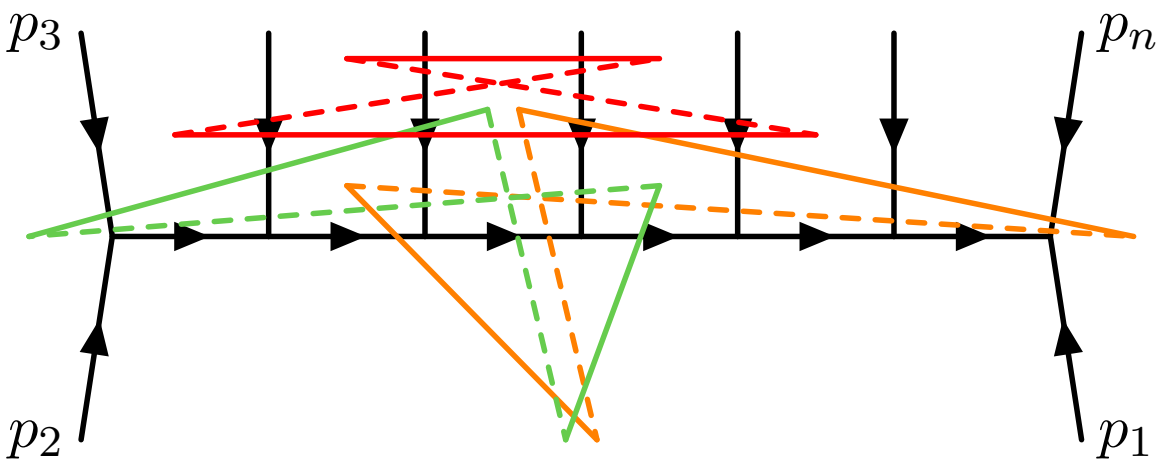
Label kinematic regions by

$$\rho = (\rho_4, \ldots, \rho_{n-1}), \qquad \rho_j = \pm 1 .$$

Winding numbers (from $s_{kl} \sim x^2_{ij}$): [Bartels, Schomerus Sprenger '14]

$$n^\rho(U_{ij}) = (\rho_{i+1} - \rho_{i+2})(\rho_{j+1} - \rho_j)/4 , \qquad 2 \le i < j \le n-1 ,$$

$$\begin{cases} n^\rho(U_{j-1,n}) = (\rho_{j+1} - \rho_j)/4 , \\ n^\rho(U_{j,1}) = -(\rho_{j+1} - \rho_j)/4 , \end{cases} \qquad 4 \le j \le n-2 .$$



Define:

$u_{ij} = U_{ij}$

$\tilde{u}_j = U_{j-1,n}/U_{j,1}$

$\varepsilon_j = U_{j-1,n} \cdot U_{j,1}$

Regions and Discontinuities

Simplest regions: Only adjacent momenta $k, \ldots, l$ flipped:

$$n^{[k,l]}(u_{ij}) = \delta_{i,k-2}\delta_{j,l}\,, \qquad n^{[k,l]}(\tilde{u}_j) = \delta_{j,k-1} - \delta_{j,l}\,.$$

$\Rightarrow$ Only one cross ratio $u_{k-2,l}$ winds.

Special case: Only one flip ($k = l$): No u_{ij} winds, but $\tilde{u}_{k-1}$, $\tilde{u}_k$ do.

Generally:

$$S[R_n] = \sum_{i,j}(u_{i,j} \otimes S_{ij}) + \sum_j(\tilde{u}_j \otimes \tilde{S}_j) + \sum_j(\varepsilon_j \otimes S_j)$$

$$\Rightarrow \quad S[\mathrm{disc}\, R_n]^\rho = 2\pi i \sum_{i,j} n^\rho(u_{ij}) S_{ij} + 2\pi i \sum_j n^\rho(\tilde{u}_j)\tilde{S}_j$$

[TB, Papathanasiou Schomerus '15]

Multi-Regge Limit Relations

Known: R_n is trivial in region $\rho = (++\ldots+)$ with no flips
This means $\mathcal{A}^{\text{BDS}}$ is correct in this region

MRL characterized by $u_{ij} \to 1$ (various rates), and $U_{j-1,n}, U_{j,1} \to 0$

Finite ratios: [Bartels, Schomerus Sprenger '14]

$$\frac{U_{j-1,n}}{1-u_{j-2,j+1}} \to \frac{1}{(1+w_j)(1+\bar{w}_j)}\,, \qquad \frac{U_{j,1}}{1-u_{j-2,j+1}} \to \frac{w_j\bar{w}_j}{(1+w_j)(1+\bar{w}_j)}\,.$$

In particular:

$$\tilde{u}_j = U_{j,1}/U_{j-1,n} \to w_j\bar{w}_j\,, \qquad \varepsilon_j = U_{j,1}/U_{j-1,n} \to 0\,.$$

Therefore:

$$0 = S[R_n]^{\text{MRL}} = \sum_j (w_j\bar{w}_j \otimes \tilde{S}_j^{\text{MRL}}) + \sum_j (\varepsilon_j \otimes S_j^{\text{MRL}})$$

That is

$$\tilde{S}_j^{\text{MRL}} = S_j^{\text{MRL}} = 0\,.$$

Region Relations

Master relations:

[TB, Papathanasiou Schomerus '15]

$$S[R_n]^\rho = 2\pi i \sum_{i,j} n^\rho(u_{ij}) S_{ij}^{\mathrm{MRL}} \,.$$

$$n^\rho(u_{ij}) = (\rho_{i+1} - \rho_{i+2})(\rho_{j+1} - \rho_j)/4 \,.$$

Relations among winding numbers for different regions:
For any set I of flipped momenta,

$$n^I(u_{ij}) = \sum_{\{k,l\}\subset I} n^{\{k,l\}}(u_{ij}) \,,$$

$$n^{\{k,l\}}(u_{ij}) = n^{[k,l]}(u_{ij}) - n^{[k+1,l]}(u_{ij}) - n^{[k,l-1]}(u_{ij}) + n^{[k+1,l-1]}(u_{ij}) \,.$$

Same relations hold for remainder function $S[R_n]^I$.

Only regions with all flips adjacent are fundamental!

Example:

$$S_8^{(-++-)} = S_8^{(----)} - S_8^{(+---)} - S_8^{(---+)} + S_8^{(+--+)} \,.$$

Taking the Limit

It remains to take the actual limit for the all-adjacent regions.

Limit:

- Symbol entries given in terms of momentum twistors
- Express in terms of Wilson loop OPE variables S_j, T_j, F_j
- Take the limit: $T_j \to 0$, with $T_j^2/S_j^2 \sim w_j\bar{w}_j$, and $F_j \sim \bar{w}_j/w_j$
- Each symbol entry becomes a unique monomial in $T_5, \ldots, T_{n-1}$
- Small cross ratios: $T_j = (\varepsilon_j^2 w_j \bar{w}_j)^{1/4}$ (recall $\varepsilon_j = U_{j-1,n}U_{j,1}$)

Manipulations:

- Extract large logs via shuffle relations
  $\log(\varepsilon_j)(x\otimes y\otimes\cdots\otimes z) = (\varepsilon_j\otimes x\otimes y\otimes\cdots\otimes z)+(x\otimes\varepsilon_j\otimes y\otimes\cdots\otimes z)$
  $+(x\otimes y\otimes\varepsilon_j\otimes\cdots\otimes z)+\cdots+(x\otimes y\otimes\cdots\otimes\varepsilon_j\otimes z)+(x\otimes y\otimes\cdots\otimes z\otimes\varepsilon_j)$
- At two loops, obtain linear and constant terms in $\log(\varepsilon_j)$
- Use symbol rule $(xy)\otimes(z) = (x)\otimes(z)+(y)\otimes(z)$ to get canonical, simplified coefficient symbols (in w_j, $\bar{w}_j$)

Systematics of All-Adjacent Regions

For a single flipped momentum, the MRL remainder function is trivial

First non-trivial case: Six points, momenta 4 & 5 flipped:

$$S[R_6^{(2)}] = \log(\varepsilon_5) f_1(w_5) + f_0(w_5) \equiv f(\varepsilon_5, w_5)$$

Seven points, momenta $4, 5, 6$ flipped:

$$S[R_7^{(2)}] = f(\varepsilon_5, v_5) + f(\varepsilon_6, v_6) + g(v_5, v_6)$$

Any number of points (checked up to $n = 10$) LLA: [Prygarin, Spradlin / Vergu, Volovich '12]

$$S[R_n^{(2)}] = \sum_{j=5}^{n-1} f(\varepsilon_j, v_j) + \sum_{j=5}^{n-2} g(v_j, v_{j+1})$$

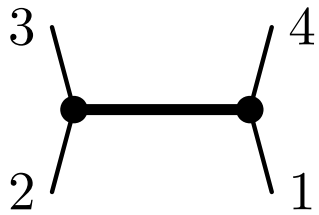
Map from w_j to v_j (found empirically)

$$v_j \equiv \frac{(1 + (1 + (\dots (1 + w_5) w_6) \dots) w_{j-1}) w_j}{1 + \left(1 + \left(\dots \left(1 + \frac{1}{w_{n-1}}\right) \frac{1}{w_{n-2}}\right) \dots\right) \frac{1}{w_{j+1}}}$$

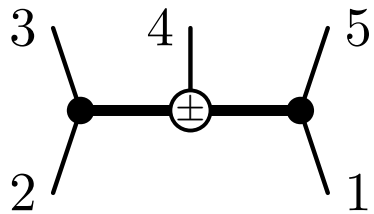
Revisit BFKL Systematics

Four-point and five-point amplitudes factorize

$$\mathcal{A}_4^{\mathrm{MRL}} = \quad = \Gamma(t) s^{\omega(t)} \Gamma(t)\,.$$



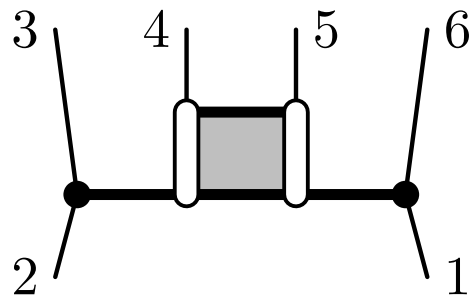
$$\mathcal{A}_5^{\mathrm{MRL}(\pm)} = \quad = \Gamma(t_4)\, s_4^{\omega(t_4)}\, \Gamma_{45}\, s_5^{\omega(t_5)}\, \Gamma(t_5)\,.$$



The six-point amplitude does not factorize!

There is a non-trivial cut contribution in the $(--)$ region:

$$\mathcal{A}_{6,(--)}^{\mathrm{MRL}} = (\text{pole terms}) + \quad .$$



[Lipatov 1008.1015] [Bartels, Kormilitzin Lipatov '13]

First sign for incompleteness of $\mathcal{A}_n^{\mathrm{BDS}}$.

Two-Reggeon states governed by integrable Hamiltonian. [Balitsky, Fadin Kuraev, Lipatov]

BFKL Systematics: Higher Points

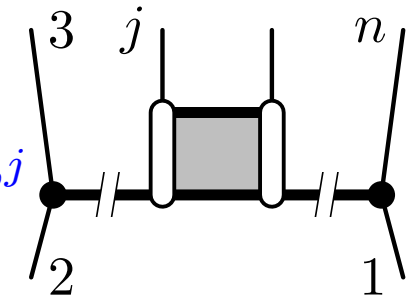
General amplitudes contain Regge pole as well as Mandelstam cut terms:

[Regge 1960] [Amati, Stanghellini Fubini 1962] [Mandelstam 1963]

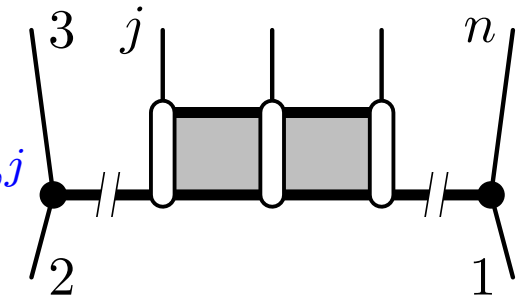
Planar $\mathcal{N} = 4$ SYM:

$$\mathcal{A}^{\mathrm{MRL}}_{n,\rho} = (\text{pole terms})^{\rho}$$

$$+\sum_j c^{\rho}_{n,1,j}$$

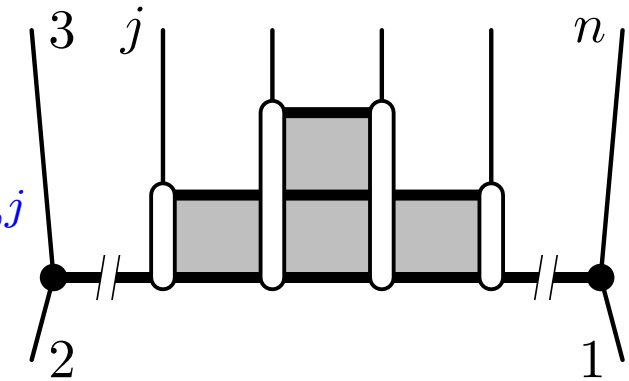


$$+\sum_j c^{\rho}_{n,2,j}$$

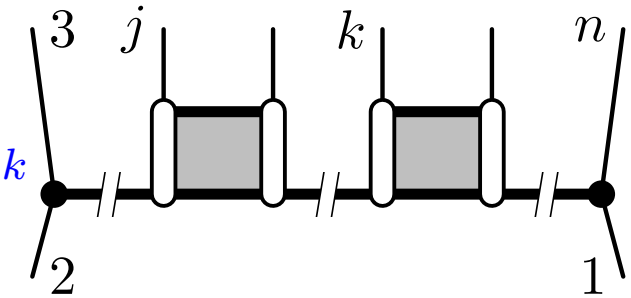


$$+\ldots$$

$$+\sum_j d^{\rho}_{n,1,j}$$



$$+\sum_j e^{\rho}_{n,j,k}$$



$$+\ldots$$

Coefficients $c, d, e, \ldots$ depend on kinematic regions

Mandelstam criterion: All-adjacent regions only see simple cuts

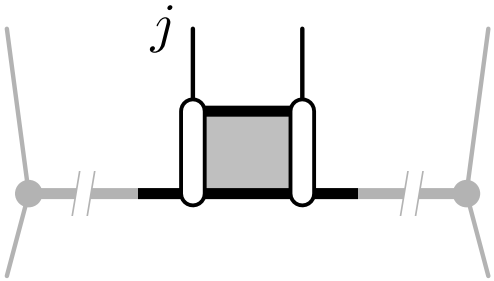
Symbol only sees simple cuts in all regions

BFKL: Remainder Function

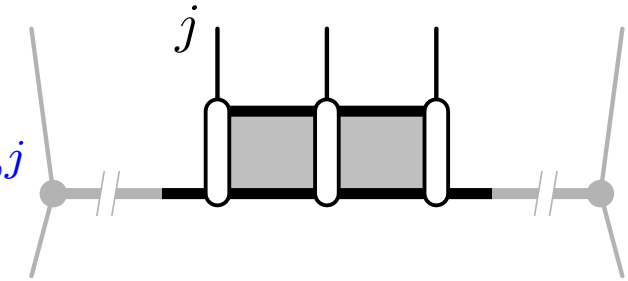
Stripping off $\mathcal{A}_n^{\text{BDS}}$, only the cut pieces remain: [Bartels, Kormilitzin Lipatov '13] [Bartels, Kormilitzin Lipatov '14]

$R_n^\rho \sim (\text{reduced pole terms})^\rho$

$$+\sum_j c^\rho_{n,1,j}$$

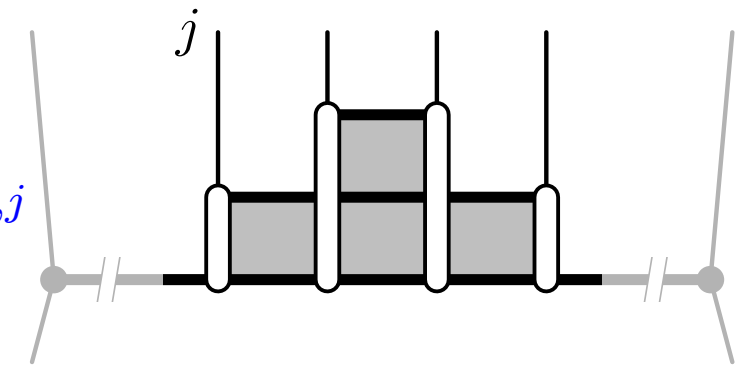


$$+\sum_j c^\rho_{n,2,j}$$

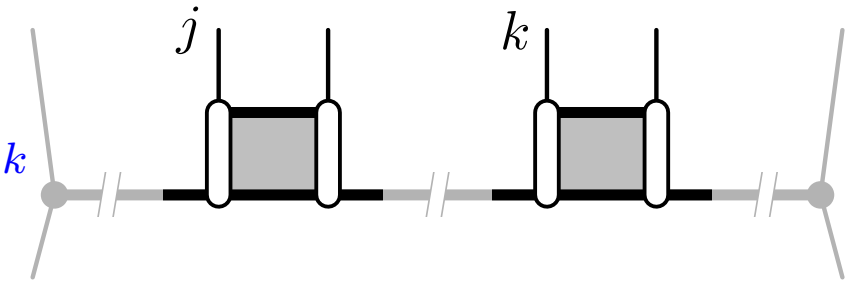


$$+\ldots$$

$$+\sum_j d^\rho_{n,1,j}$$



$$+\sum_{j<k} e^\rho_{n,j,k}$$



$$+\ldots$$

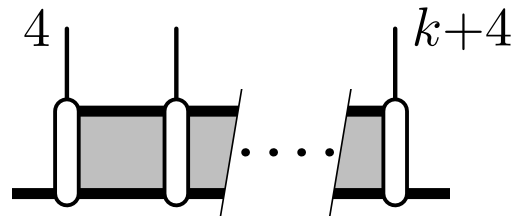
BFKL: Cut Integrals

Multi-Reggeon bound states governed by BFKL [Balitsky, Fadin Kuraev, Lipatov] and BKP equations. [Bartels 1980] [Kwieciński Praszałowicz]

Solutions expressed in terms of $\mathrm{SL}(2,\mathbb{C})$ representation labels n_i, ν_i.

General cut term spanning k t-channels: [Bartels, Kormilitzin Lipatov '14]

$$f_k(\varepsilon_5,\ldots,\varepsilon_{k+4};w_5,\ldots,w_{k+4}) \equiv$$



$$= i\,g \sum_{n_5,\ldots,n_{k+4}} \int \mathrm{d}\nu_5 \ldots \mathrm{d}\nu_{k+4}\, \Phi_{\mathrm{L},5}\, \varepsilon_5^{gE_{n_5,\nu_5}}\, C_5 \cdot$$

$$\cdot\, \varepsilon_6^{gE_{n_6,\nu_6}}\, C_6 \ldots C_{k+3}\, \varepsilon_{k+4}^{gE_{n_{k+4},\nu_{k+4}}}\, \Phi_{\mathrm{R},k+4}\,.$$

Sum over states labeled by (n_j,ν_j). BFKL "data":

- Impact factors Φ_{L}, Φ_{R}
- Green's functions $\varepsilon^{gE_{n,\nu}}$ with BFKL eigenvalue (energy) $E_{n,\nu}$
- Central emission blocks C

BFKL: Kinematic Dependence

The impact factors $\Phi_{\mathrm{L},5}$, $\Phi_{\mathrm{R},k+4}$, and emission blocks C_j contain kinematic factors that combine into Fourier–Mellin transformation kernels

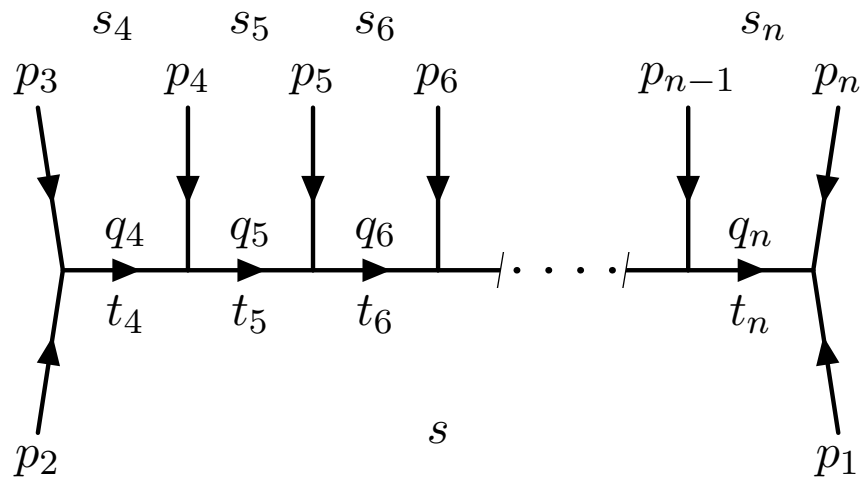
[Bartels, Kormilitzin, Lipatov, Prygarin]

$$w_k^{i\nu_k+n_k/2}\,\bar{w}_k^{i\nu_k-n_k/2} = \rho_k^{2i\nu_k}\,e^{in_k\varphi_k} \qquad \text{for} \qquad w_k = \rho_k e^{i\varphi_k}\,.$$

Here,

$$w_j = \frac{p_{j-1}q_{j+1}}{q_{j-1}p_j}\,, \qquad j = 5,\ldots,n-1$$

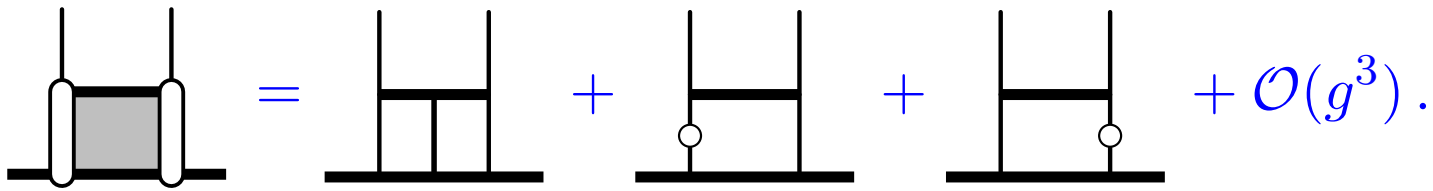
are anharmonic ratios of complex p_j, q_j that parametrize the momenta and t-channel momentum transfers in the 2d plane transverse to p_1, p_2.



These are the same as the previous w_j, $\bar{w}_j$ parameters!

BFKL: Perturbative Expansion, Six Points

Expand simplest cut to two loops (six points):



Rung in first term stems from g^1 term in BFKL Green's function

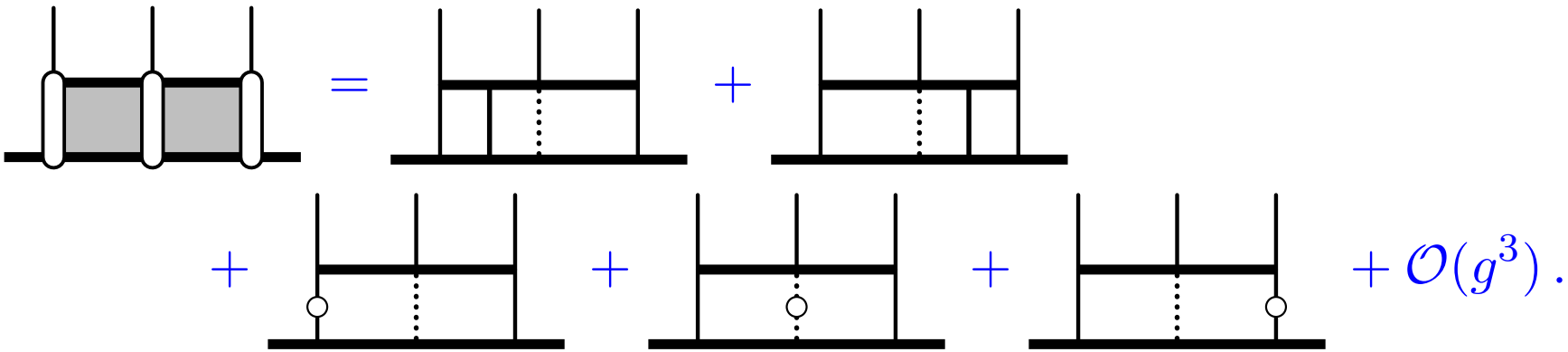
$$G_2(n_k, \nu_k) = \varepsilon_k^{gE_{n_k,\nu_k}} = 1 + g\, E^{(0)}_{n_k,\nu_k} \log(\varepsilon_k) + \mathcal{O}(g^2)\,,$$

it provides the large logarithm $\log(\varepsilon_k)$. Green's functions are the only source of ε_k dependence, everything else solely depends on $w_j, \bar{w}_j$.

Further terms are subleading in logarithms (NLLA)

BFKL: Perturbative Expansion, Seven Points

Seven-point cut term to two loops:



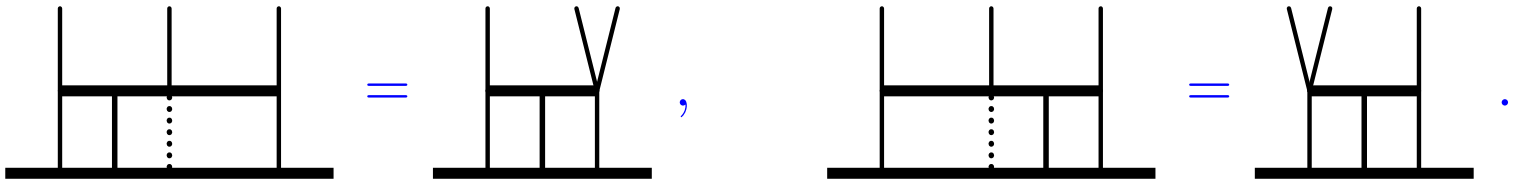
- First appearance of central emission block
- First two terms are leading logarithmic (LLA)
- Last three terms are NLLA

Crucial reduction identities:

[Bartels, Kormilitzin, Lipatov, Prygarin]

BFKL: Seven-Point Reduction

Use identities to reduce LLA pieces to six points:



Variables:

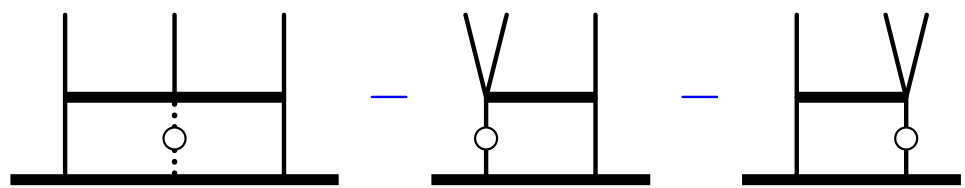
$$v_{5,6;5} \equiv \frac{p_4 q_7}{q_4(p_5+p_6)} = \frac{w_5}{(1+\frac{1}{w_6})}\,, \qquad v_{5,6;6} \equiv \frac{(p_4+p_5)q_7}{q_4 p_6} = (1+w_5)w_6\,.$$

Split off six-point pieces (including NLLA):

$$f_{2,(2)}(\varepsilon_5,\varepsilon_6;w_5,w_6) = f_{1,(2)}(\varepsilon_5;v_{5,6;5}) + f_{1,(2)}(\varepsilon_6;v_{5,6;6}) + g_2(v_{5,6;5},v_{5,6;6})\,,$$

with

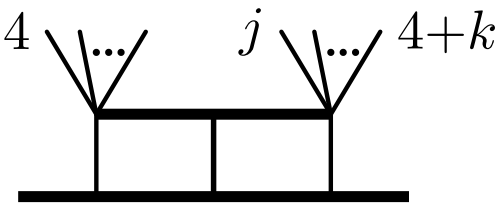
$$g_2(v_{5,6;5},v_{5,6;6}) =$$



BFKL: Reduction for All Multiplicities

Any n, leading logarithmic approximation: see also: [Prygarin, Spradlin / Vergu, Volovich '12]

$$f^{\mathrm{LLA}}_{k,(2)}(\varepsilon_5,\ldots,\varepsilon_{4+k};w_5,\ldots,w_{4+k}) = \sum_{j=5}^{4+k}$$



Include NLLA terms:

$$f_{k,(2)}(\varepsilon_5,\ldots,\varepsilon_{4+k};w_5,\ldots,w_{4+k}) = \sum_{j=5}^{4+k} f_{1,(2)}(\varepsilon_j;v_j) + \sum_{j=5}^{3+k} g_2(v_j,v_{j+1})\,,$$

Variables:

$$v_{k,l;j} \equiv \frac{q_{k-1}-q_j}{q_{k-1}}\,\frac{q_{l+1}}{q_j-q_{l+1}} = \frac{(1+(1+(\ldots(1+w_k)w_{k+1})\ldots)w_{j-1})w_j}{1+\left(1+\left(\ldots\left(1+\frac{1}{w_l}\right)\frac{1}{w_{l-1}}\right)\ldots\right)\frac{1}{w_{j+1}}}$$

Exactly recover previous result!

Two-Loop Summary

From symbol: Have found rather simple formula for symbol in all kinematic regions, containing only six-point and seven-point data. Result for any number of points is sum over six-point and seven-point building blocks. Somewhat heuristic.

BFKL systematics: Previous finding reproduced by two-loop expansion of relevant Regge cut integral. Origin of previously found variables v_j understood.

Implication: Actually, findings from symbol imply that all prefactors $c_{n,b,j}$ of simple cut terms are identical! Those are very cumbersome to fix by conventional means.

NLO emission block: The function g_2 contains information on the NLO emission block, which is not known to date. Could in principle be extracted by inverting the Fourier–Mellin transform.

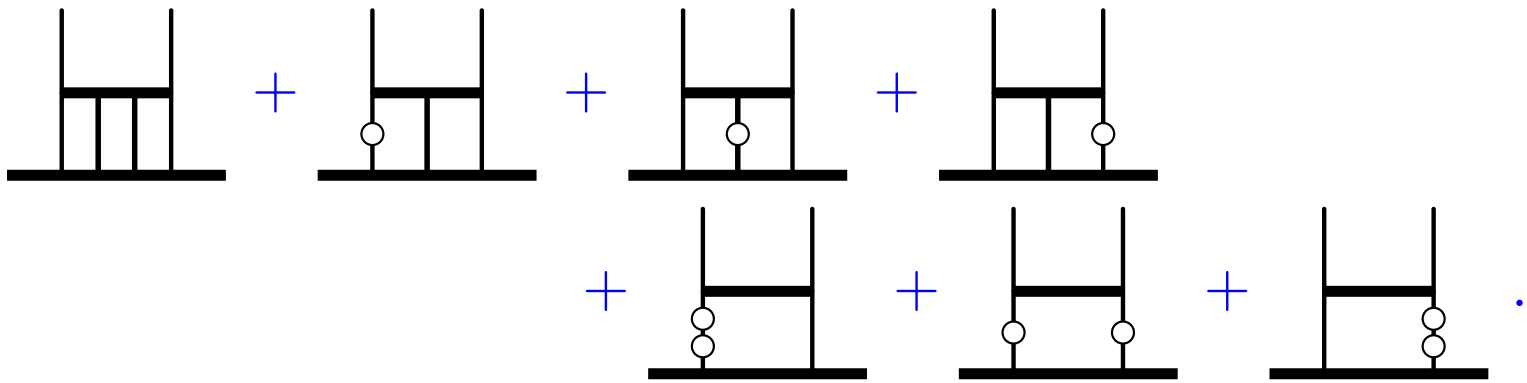
BFKL: Three-Loop Expansion

Expansion of simple two-Reggeon cut diagram at three loops:
Judiciously arrange all terms $\rightarrow$ five building blocks, [TB 1606.07640]

$$
\begin{aligned}
f_{k,(3)}(\varepsilon_5,\ldots,\varepsilon_{k+4};w_5,\ldots,w_{k+4}) &= \sum_{j=5}^{k+4} f_{1,(3)}(\varepsilon_j;v_{5,4+k;j}) \\
&+ \sum_{i=5}^{k+3}\sum_{j=i+1}^{k+4} g_3(\varepsilon_i,\varepsilon_j;v_{5,j-1;i},v_{i+1,k+4;j}) \\
&+ \sum_{i=5}^{k+2}\sum_{j=i+1}^{k+3} g_{\mathrm{L}}(\varepsilon_i;v_{5,j-1;i},v_{i+1,j;j},v_{j+1,k+4;j+1}) \\
&+ \sum_{i=5}^{k+2}\sum_{j=i+2}^{k+4} g_{\mathrm{R}}(\varepsilon_j;v_{5,i;i},v_{i+1,j-1;i+1},v_{i+2,k+4;j}) \\
&+ \sum_{i=5}^{k+1}\sum_{j=i+2}^{k+3} h(v_{5,i;i},v_{i+1,j-1;i+1},v_{i+2,j;j},v_{j+1,k+4;j+1})\,.
\end{aligned}
$$

BFKL: Three-Loop Building Blocks I

Six-point data (starts at LLA):

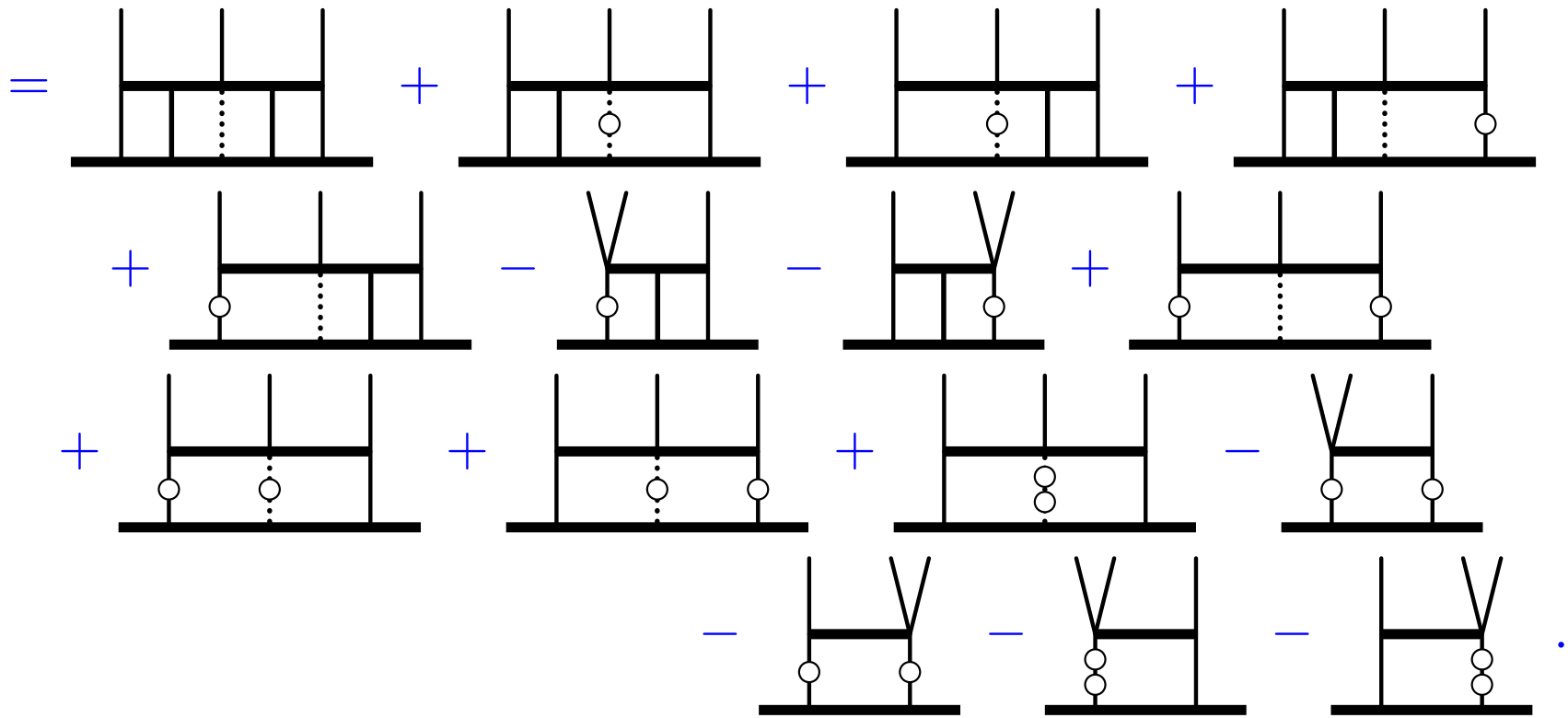
$f_{1,(3)} =$ 

BFKL: Three-Loop Building Blocks II

Seven-point data (starts at LLA):

$$g_3(\varepsilon_5, \varepsilon_6; w_5, w_6)$$

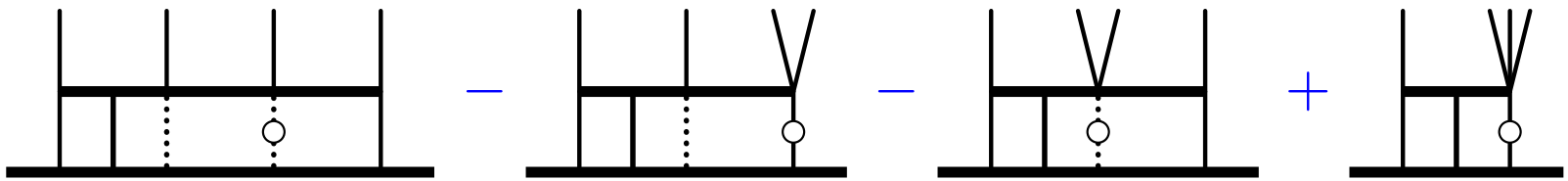
$$\equiv f_{2,(3)}(\varepsilon_5, \varepsilon_6; w_5, w_6) - f_{1,(3)}(\varepsilon_5; v_{5,6;5}) - f_{1,(3)}(\varepsilon_6; v_{5,6;6})$$



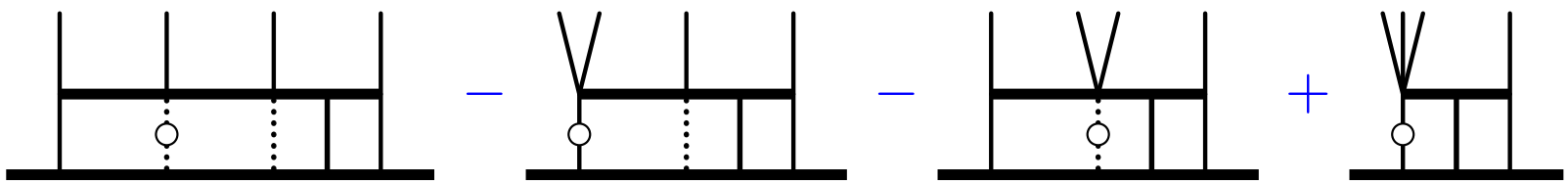
BFKL: Three-Loop Building Blocks III

NLLA building blocks (appear at 8 points):

$g_{\mathrm{L}}(\varepsilon_5; w_5, w_6, w_7)$

$=$  ,

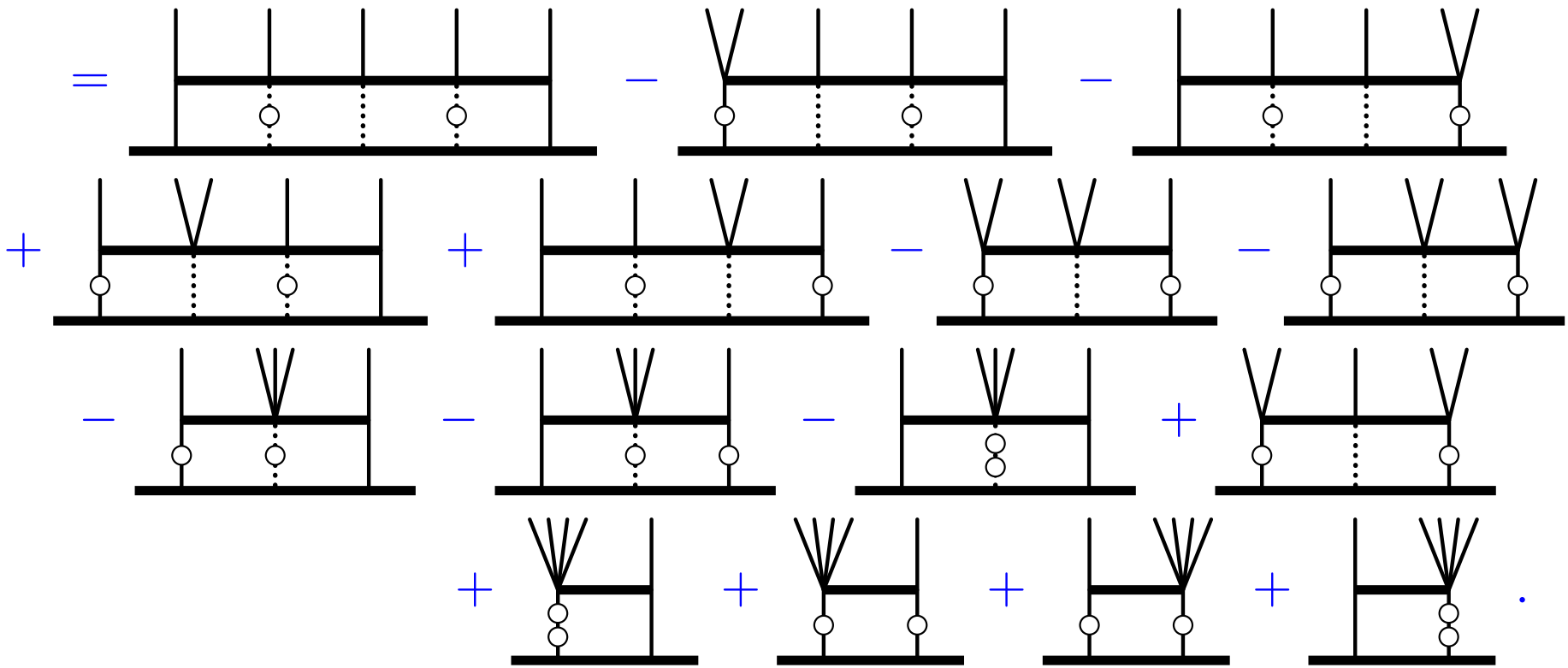
$g_{\mathrm{R}}(\varepsilon_7; w_5, w_6, w_7)$

$=$  ,

BFKL: Three-Loop Building Blocks IV

NNLLA building block (appears at 9 points):

$h(w_5, w_6, w_7, w_8)$



Three Loops: Building Blocks Versus Data

The available six-point and seven-point three-loop symbols suffice for the building blocks $f_{1,(3)}$ and g_3 that start contributing at LLA.

New information appears at NLLA and eight points, which is not covered by available data. Eight points: No known data on general kinematics beyond the two-loop symbol. Difficult!

Beyond that, there is another (NNLLA) building block that starts appearing at nine points.

If the NLO central emission block would be known, the NLLA building blocks g_{L} and g_{R} could in principle be computed from the integral representation. Information on the NLO emission block is contained in the two-loop building block g_2.

In turn, could extract NNLO emission block from seven-point three-loop building block g_3.

Summary of Systematics

The multi-Regge limit admits to "recycle" data from low multiplicities to construct amplitudes for all multiplicities. The reason is exponentiation and factorization. Once the BFKL building blocks (eigenvalues, impact factors, emission blocks) are known to a certain order, one can construct all amplitudes to that order.

At fixed loop order: Completely understood how to construct amplitudes for any number of points once the data for enough points is known. Explicitly done for two and three loops (four loops: partly).

The known data contains information on the central emission block at NLO (two loops) and NNLLO (three loops). One can in principle extract those by subtracting all other terms and inverting the Fourier–Mellin transform.

Directions

Possible directions:

- Extract information on higher Regge cuts from available data? (More Reggeons, more general structure.)
- For simple two-Reggeon cuts, the available perturbative data seems exhausted: It is not possibel to go beyond LLA at three or more loops with the available data. Generating higher-point data ($n \geq 8$) for general kinematics will be very difficult.
- Make use of the integrability-based Wilson loop OPE? It is based on the collinear limit. How much information is contained in the overlap of the collinear and multi-Regge limit? Has partly been investigated.

  [Hatsuda 1404.6506] [Basso, Caron-Huot Vieira 2014]
- Understand the relation between BFKL integrability and integrable structures for scattering amplitudes in general kinematics.

Outlook

Want to understand the principles QFT as much as possible. Eventually want a UV completion. One route: First, completely understand one particular theory. Best candidate: $\mathcal{N} = 4$ super Yang–Mills theory (planar limit). Want to solve the theory: Amplitudes as functions of finite g (all values).

Appears possible (with some luck) due to integrability.

For amplitudes, the Wilson loop OPE provides an integrability-based framework, but is restricted to the collinear limit.

The multi-Regge limit is another limit that is known to benefit from integrable structures. Are amplitudes solvable at finite g in MRL?

First step: Extract as much information as possible from overlap of MRL with collinear limit. Work in progress.