Integrable Deformations for $\mathcal{N} = 4$ Super Yang–Mills and ABJM Amplitudes

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Abstract

Integrands for scattering amplitudes admit deformations that preserve the integrable symmetries. I will explain these deformations in terms of Graßmannian integrals and on-shell diagrams, and discuss possible applications. This is a slightly extended version of a talk that I gave at the Simons Center in Stony Brook on November 11, 2014.

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Contents

1 Prelude

This talk is based on [1],[1] which builds upon earlier work [3,4]. In this talk, I will exclusively consider the planar limit of $\mathcal{N} = 4$ super Yang–Mills and ABJM theory.

2 Graßmannian Integral and On-Shell Diagrams

Before explaining the deformations, I will give a mini-review of the Graßmannian integral and on-shell diagrams [5], as these are central to everything below.

The Graßmannian Integral. The Graßmannian integral $\mathcal{G}_{n,k}$ is defined as an integral over the Graßmannian space of k-planes in n-dimensional complex space,

$$\mathcal{G}_{n,k}(\mathcal{W}_i) = \int \frac{\mathrm{d}^{k\cdot n}C}{|\mathrm{GL}(k)|} \frac{\delta^{4k|4k}(C \cdot \mathcal{W})}{M_1 \cdot \dots \cdot M_n}\,, \tag{2.1}$$

where the twistor variables $\mathcal{W}_i$ parametrize the external states.[2] In $(2,2)$ signature, the twistors are given by $\mathcal{W}_i = (\tilde{\mu}_i, \tilde{\lambda}_i, \tilde{\eta}_i)$, where $\tilde{\mu}_i$ is the Fourier transform of λ_i, and λ_i, $\tilde{\lambda}_i$ parametrize the momentum, $p_i^{a\dot{a}} = \lambda_i^a \tilde{\lambda}_i^{\dot{a}}$. The Graßmann variables $\tilde{\eta}_i$ parametrize the superfield Φ_i. The denominator consists of minors $M_i = |C_i, \dots, C_{i+k-1}|$ of the matrix C. Converting back to conventional spacetime variables, the delta functions become

$$\delta^{4k|4k}(C \cdot \mathcal{W}) \longrightarrow \delta^{2(n-k)}(C^{\mathsf{T}} \cdot \lambda)\, \delta^{2k}(C \cdot \tilde{\lambda})\, \delta^{4k}(C \cdot \tilde{\eta})\,. \tag{2.2}$$

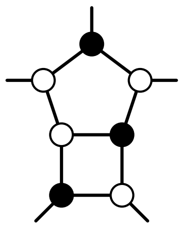
[1] Note also [2], which has some overlap with [1].

[2] In [1] the conventional twistors are denoted by $\mathcal{Z}_i$, and the momentum twistors are denoted by $\mathcal{W}_i$. Here we use the converse notation. Both conventions are used in the literature.

The integral (2.1) is to be interpreted as a multidimensional contour integral. After localizing all but four of the bosonic delta functions (the remaining four will become the momentum conservation delta functions), $k(n-k)-(2n-4)=n(k-2)-k^2+4$ integrations remain, which are supposed to be localized on zeros of the minors M_i by the residue theorem.

The Graßmannian integral looks rather innocent, but in fact contains a wealth of information. By picking the right integration contour (poles), it generates all tree-level amplitudes of $\mathcal{N}=4$ super Yang–Mills theory, as well as all leading singularities of loop amplitudes in this theory.

On-Shell Diagrams. The individual residues of the Graßmannian integral can be identified as on-shell diagrams. These are planar graphs with trivalent vertices of two types: On-shell three-particle MHV amplitudes (black dots), and on-shell three-particle anti-MHV amplitudes (white dots). The vertices are connected to each other by internal lines, which indicate that the respective on-shell variables should be identified and integrated over, with the integration measure $\mathrm{d}^2\lambda\,\mathrm{d}^2\tilde{\lambda}\,\mathrm{d}^4\tilde{\eta}/|\mathrm{GL}(1)|$. Some of the lines on the vertices remain as external lines. As an example, this is the diagrammatic form of the five-point MHV amplitude:



(2.3)

At tree level, on-shell diagrams provide the individual terms in BCFW expansions of the amplitude. At loop level, *reduced* on-shell diagrams (which can be obtained from the Graßmannian integral) form the leading singularities of loop amplitudes. Unreduced diagrams (which cannot be obtained from the Graßmannian integral) can be used to construct the complete loop integrand at any loop order [5].

The reformulation of scattering amplitudes in terms of the Graßmannian integral and on-shell diagrams is a great and beautiful story that has fascinating relations to topics of current interest to mathematicians in projective geometry and combinatorics. In particular, it provides a direct (recursive) construction of the complete planar loop integrand to any loop order, without making any reference to Feynman diagrams or gauge symmetry.

Caveat. But there is a caveat. *There is currently no known practical way to integrate the integrand.* This is of course not unexpected, as $\mathcal{N}=4$ SYM is a massless, conformal theory, and hence scattering amplitudes are not well-defined. Conventionally, one employs dimensional regularization to get a well-defined result. However, doing so would require us to translate the integrand back to the "conventional" space-time description. But then all the beautiful structure would get lost, and nothing would be gained. In particular, dimensional regularization breaks the conformal symmetry. But even for quantities that are known to be finite in exactly four dimensions, such as the ratio function $\mathcal{R}_n=\mathcal{A}_n/\mathcal{A}_n^{\mathrm{MHV}}$, we lack an integration procedure that manifestly cancels all the divergences among the individual terms and allows to actually perform the integration.

This provides the motivation to study deformations of the Graßmannian integral and/or the on-shell diagrams. Ideally, such a deformation would preserve as much of the symmetries of these quantities as possible.

3 Symmetries

Scattering amplitudes, the Graßmannian integral, and on-shell diagrams are functions of n twistor variables $\mathcal{W}_i$. The symmetries of these objects are:

- Superconformal symmetry. The superconformal symmetry group of $\mathcal{N} = 4$ SYM is $\mathfrak{psu}(2,2|4)$, and in twistor variables, its generators take the form

$$\mathfrak{J}^{\mathcal{A}}{}_{\mathcal{B}} = \sum_{i=1}^{n} \mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{B}}} - (\text{supertrace})\,. \tag{3.1}$$

- Dual superconformal symmetry [6]. This is an entire additional copy of $\mathfrak{psu}(2,2|4)$ which, in twistor variables, acts in a very non-local representation.[3]

- The conventional and the dual superconformal symmetry close into $\mathrm{Y}(\mathfrak{psu}(2,2|4))$, which is an infinite-dimensional Yangian symmetry algebra [7]. The Yangian algebra is organized into infinitely many *levels*. Level zero consists of the conventional $\mathfrak{psu}(2,2|4)$ superconformal symmetry. The level-one generators take the general form

$$\widehat{\mathfrak{J}}^{a} = f^{a}{}_{bc} \sum_{\substack{i,j=1\\ i<j}}^{n} \mathfrak{J}_i^{b}\,\mathfrak{J}_j^{c} + \sum_{i=1}^{n} u_i\,\mathfrak{J}_i^{a}\,, \tag{3.2}$$

  where $\mathfrak{J}_i^a$ are the level-zero generators acting at site i.[4] The second term is not present in the representation for the undeformed amplitudes, but it will appear for the deformed amplitudes. All higher-level generators are obtained from the first two levels by iterated commutators (modulo the Serre relations).

4 Deformations: 4d $\mathcal{N} = 4$ SYM

The study of the deformations I will discuss initially was motivated by the observation of B. Zwiebel that the tree-level S-matrix equals a specific piece of the spin-chain dilatation operator [8]. For the dilatation generator, there is a construction in terms of an R-matrix. The R-matrix in particular depends on the spectral parameter, which is central to integrability. The motivation to study deformations is to introduce something akin to the spectral parameter to the scattering amplitude problem. The hope is that one could do analysis in this new parameter, and extract new information about scattering amplitudes. Perhaps this would lead to a regulated, integrated amplitude.

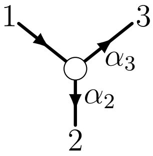
In the following, I will mostly review earlier results on deformations [3, 4]. We want to study deformations that preserve a maximal amount of symmetry. Ideally, they should preserve the complete infinite-dimensional Yangian symmetry. One can start with the simplest building

[3] The dual superconformal symmetry becomes local in momentum-twistor coordinates $\mathcal{Z}_i = (\lambda_i, \mu_i, \xi_i)$, where $\mu_i^{\dot{a}} = x_i^{a\dot{a}}\varepsilon_{ab}\lambda_i^b$, $\xi_i^A = \theta_i^{Aa}\varepsilon_{ab}\lambda_i^b$, and $x_i - x_{i+1} = p_i$, $\theta_i - \theta_{i+1} = \lambda_i\tilde{\eta}_i$.

[4] For the linear level-zero representation (3.1), the bilocal combinations in the level-one generators take the form $f^{a}{}_{bc}\,\mathfrak{J}_i^b\,\mathfrak{J}_j^c = (-1)^{\mathcal{C}}\,\mathfrak{J}_i^{\mathcal{A}}{}_{\mathcal{C}}\mathfrak{J}^{\mathcal{C}}{}_{\mathcal{B}} - (i \leftrightarrow j)$.

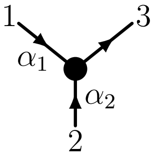
blocks, the tree-level three-point amplitudes[5]

$$\widehat{\mathcal{A}}_3^{\circ} =$$



$$= \int \frac{\mathrm{d}\alpha_2}{\alpha_2^{1+a_2}} \frac{\mathrm{d}\alpha_3}{\alpha_3^{1+a_3}}\, \delta^{4|4}(C_\circ \cdot \mathcal{W}) \simeq \frac{\delta^4(P)\,\delta^4(\tilde{Q})}{[12]^{1+a_3}[23]^{1-a_2-a_3}[31]^{1+a_2}}\,,$$

$$\widehat{\mathcal{A}}_3^{\bullet} =$$



$$= \int \frac{\mathrm{d}\alpha_1}{\alpha_1^{1+a_1}} \frac{\mathrm{d}\alpha_2}{\alpha_2^{1+a_2}}\, \delta^{8|8}(C_\bullet \cdot \mathcal{W}) \simeq \frac{\delta^4(P)\,\delta^8(Q)}{\langle 12\rangle^{1-a_1-a_2}\langle 23\rangle^{1+a_1}\langle 31\rangle^{1+a_2}}\,, \tag{4.1}$$

where

$$C_\circ = \begin{pmatrix} 1 & \alpha_2 & \alpha_3 \end{pmatrix}, \qquad C_\bullet = \begin{pmatrix} 1 & 0 & \alpha_1 \\ 0 & 1 & \alpha_2 \end{pmatrix}, \tag{4.2}$$

and $\mathcal{W}_i^A$ are twistor variables that parametrize the external states. By direct inspection, one can show that these vertices are invariant under the Yangian symmetry with evaluation parameters $\{u_1, u_2, u_3\}$ provided that

$$\begin{aligned} \widehat{\mathcal{A}}_3^{\circ}:\quad & u_1^+ = u_3^-\,, \quad u_2^+ = u_1^-\,, \quad u_3^+ = u_2^-\,, \\ \widehat{\mathcal{A}}_3^{\bullet}:\quad & u_1^+ = u_2^-\,, \quad u_2^+ = u_3^-\,, \quad u_3^+ = u_1^-\,, \end{aligned} \tag{4.3}$$

where $u_i^{\pm} = u_i \pm c_i$, and c_i are the local central charges, that is they are the eigenvalues of the local central charge generator

$$\mathfrak{C}_i = -\mathcal{W}_i^C \frac{\partial}{\partial \mathcal{W}_i^C}\,. \tag{4.4}$$

In terms of the deformation parameters a_i, they read

$$\begin{aligned} \widehat{\mathcal{A}}_3^{\circ}:\quad & c_1 = a_2 + a_3\,, \quad c_2 = -a_2\,, \quad c_3 = -a_3\,, \\ \widehat{\mathcal{A}}_3^{\bullet}:\quad & c_1 = a_1\,, \qquad\quad\; c_2 = a_2\,, \quad\;\; c_3 = -a_1 - a_2\,. \end{aligned} \tag{4.5}$$

Deformed higher-point diagrams can be obtained by iteratively gluing three-point vertices using on-shell integration. Whenever one glues two invariants, or glues two external lines of a single invariant, the result will again be invariant, as long as the evaluation parameters u, u' and central charges c, c' on the glued lines satisfy [4]

$$u = u'\,, \qquad c = -c'\,. \tag{4.6}$$

Iterating this procedure, one can construct deformed Yangian-invariant versions of all on-shell diagrams. Combining the conditions (4.3) with the gluing conditions (4.6), the parameters of all deformed diagrams must satisfy

$$\boxed{u_i^+ = u_{\sigma(i)}^-\,,} \tag{4.7}$$

where σ is the permutation that is associated to the diagram. It is obtained from the diagram by following the "left-right paths" through the diagram, turning right at each black (MHV) vertex, and left at each white ($\overline{\text{MHV}}$) diagram. For example, the permutation associated to the five-point MHV diagram (2.3) is $\{3, 4, 5, 1, 2\}$.[6]

[5] Here, $[ij] \equiv \varepsilon_{\dot\alpha\dot\beta}\tilde\lambda_i^{\dot\alpha}\tilde\lambda_j^{\dot\beta}$, $\langle ij\rangle \equiv \varepsilon_{\alpha\beta}\lambda_i^{\alpha}\lambda_j^{\beta}$, $P \equiv \sum_{i=1}^n \lambda_i\tilde\lambda_i$, $Q \equiv \sum_{i=1}^n \lambda_i\eta_i$, and $\tilde{Q} \equiv ([12]\eta_3 + [23]\eta_1 + [31]\eta_2)$.

[6] Tree-level MHV amplitudes are top-cell diagrams with $k = 2$. In general, all top-cell diagrams are characterized by permutations that are cyclic shifts by k sites.

In summary, we have obtained Yangian-invariant deformations for all on-shell diagrams. Every deformed diagram is characterized by n central charges $c_{1\dots n}$ and n evaluation parameters $u_{1\dots n}$, subject to n constraints (4.7). In the undeformed case, the central charge operator (4.4) is related to the helicity via

$$\mathfrak{h}_i = 1 - \mathfrak{C}_i \,. \tag{4.8}$$

Thus one could interpret the deformations as deformations of the helicities of the external states.

Note that the above deformations at this stage are purely formal mathematical constructions, and do not constitute scattering amplitudes for any known theory. (And perhaps such a theory is unlikely to exist.) Before discussing the possible use of these deformations, I want to discuss the ABJM case, whose diagrams resemble conventional integrable systems more closely than the $\mathcal{N} = 4$ SYM diagrams.

5 Deformations: 3d ABJM

For all the nice structures of 4d $\mathcal{N} = 4$ SYM, counterparts have been found in the three-dimensional ABJM theory. So far, the ABJM avatars have usually been (technically) more complicated than in the $\mathcal{N} = 4$ case. Here, we finally have a situation where the analysis is technically simpler in ABJM theory than in $\mathcal{N} = 4$ SYM.

Also in ABJM theory, there exists a Graßmannian integral [9], and on-shell diagrams [10, 11] that can be combined to form the tree-level amplitudes and loop integrands of the theory. Hence we can try to deform these structures in a similar way as in the 4d case.

The symmetry algebra of ABJM amplitudes is the Yangian $\mathrm{Y}(\mathfrak{osp}(6|4))$. In contrast to the four-dimensional case, the little group of massless momenta is trivial in three dimensions, and hence there is no notion of helicity, and there are no local central charges that could be deformed. Nevertheless, deformations are possible.

5.1 Deformed On-Shell Diagrams

The simplest amplitude in ABJM theory is the four-point amplitude, and it can be deformed as follows:

$$\widehat{\mathcal{A}}_4(z) = \int \frac{\mathrm{d}\theta}{\sin(\theta)^{1+z}} \delta^{4|6}\Big(C(\theta) \cdot \Lambda\Big) = \frac{\delta^3(P)\,\delta^6(Q)}{\langle 12\rangle^{1-z}\langle 23\rangle^{1+z}} \,, \tag{5.1}$$

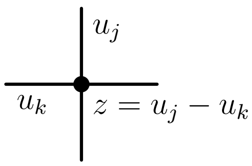
where $\Lambda^{\mathcal{A}} = (\lambda^a, \eta^A)$ parametrizes the external on-shell superfields, and

$$C(\theta) = \begin{pmatrix} 1 & 0 & i\cos(\theta) & i\sin(\theta) \\ 0 & 1 & -i\sin(\theta) & i\cos(\theta) \end{pmatrix} . \tag{5.2}$$

This deformed amplitude is invariant under the Yangian generators with evaluation parameters $\{u_1, \dots, u_4\}$, provided that[7]

$$u_1 = u_3 \,, \qquad u_2 = u_4 \,, \qquad z = u_1 - u_2 \,. \tag{5.3}$$

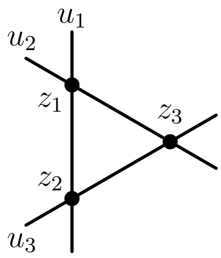
Pictorially, the fundamental ABJM four-vertex is



. (5.4)

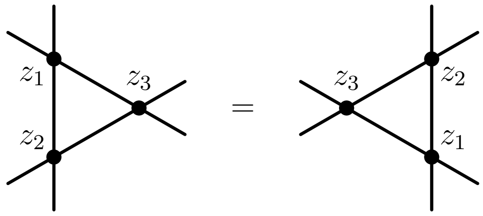
[7]Actually, $z = \pm(u_1 - u_2)$, where the sign depends on which columns in the matrix $C(\theta)$ are set to the unit matrix. The sign will not be essential here and is thus omitted.

Again, higher-point diagrams can be obtained by iteratively gluing four-point vertices together by on-shell integration. Yangian invariance is preserved as long as the evaluation parameters u, u' associated to the two glued lines are identical. Thus every deformed $2k$-point diagram will depend on k independent evaluation parameters u_i, and the parameters z_i at the vertices are differences of the evaluation parameters associated to the two lines that cross at the respective vertex. Here is a simple example:



$$z_1 = u_1 - u_2\,,\qquad z_2 = u_1 - u_3\,,\qquad z_3 = u_2 - u_3\,. \tag{5.5}$$

Diagrams of this type look reminiscent of vertex models, with vertex parameters, and rapidity variables on each line. Also, these diagrams satisfy a triangle inequality:



, (5.6)

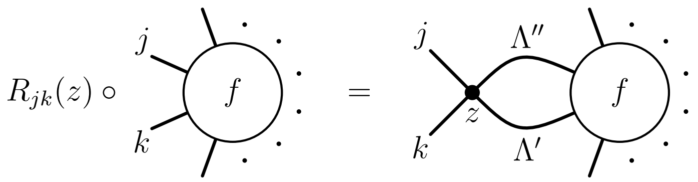
which holds for any triangle of three vertices within a bigger diagram. This triangle equality looks exactly like a Yang–Baxter equation. As we shall see, this is not a coincidence.

5.2 R-Matrix Construction

We can reformulate the above in terms of an R-matrix, which makes the above construction very reminiscent of conventional integrable models. Define the operator

$$(R_{jk}(z)\circ f)(\dots,\Lambda_j,\Lambda_k,\dots) \equiv \int \mathrm{d}\Lambda'\,\mathrm{d}\Lambda''\,\mathcal{A}_4(z)(\Lambda_j,\Lambda_k,i\Lambda',i\Lambda'')\,f(\dots,\Lambda'',\Lambda',\dots)\,. \tag{5.7}$$

Pictorially:



. (5.8)

By the explicit form of the four-vertex (5.1), the R-operator acts by integrating over a weighted rotation of two external legs,

$$(R_{jk}(z)\circ f)(\Lambda) \equiv \int \frac{\mathrm{d}\theta}{\sin(\theta)^{1+z}}\,f(\Lambda)\Big|_{\binom{\Lambda_j}{\Lambda_k}\,\to\, i\begin{pmatrix}\cos(\theta) & \sin(\theta)\\ \sin(\theta) & -\cos(\theta)\end{pmatrix}\binom{\Lambda_j}{\Lambda_k}} \tag{5.9}$$

By the above discussion of deformed on-shell diagrams, acting with the R-operator is equivalent to gluing the four-vertex to an invariant, and hence preserves invariance (up to an exchange of the two evaluation parameters). That is

$$\widehat{\mathfrak{J}}^a(\dots,u_j,u_k,\dots)\,R_{jk}(z) = R_{jk}(z)\,\widehat{\mathfrak{J}}^a(\dots,u_k,u_j,\dots), \tag{5.10}$$

when acting on Yangian-invariant functions.

We can further define a Lax operator

$$L_i(u) \equiv u\mathbf{1} + \sum_a \mathfrak{J}_i^a\, \boldsymbol{e}_a\,, \tag{5.11}$$

where $\mathfrak{J}_i^a$ are the level-zero generators of $\mathfrak{osp}(6|4)$ acting on the external state i, and $\boldsymbol{e}_a$ denotes the generators of the fundamental representation. The Lax operator gives rise to a monodromy operator

$$T(u_0, \vec{u}) \equiv L_1(u_0 - \tfrac{1}{2}u_1)L_2(u_0 - \tfrac{1}{2}u_2)\dots L_{2k}(u_0 - \tfrac{1}{2}u_{2k}). \tag{5.12}$$

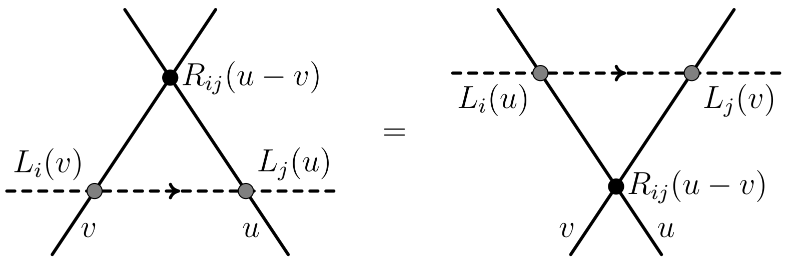
By standard procedure,[8] expanding the monodromy yields the Yangian generators:

$$T(u_0, \vec{u}) = u_0^{2k} + u_0^{2k-1}\mathfrak{J}^{(0)}(\vec{u}) + u_0^{2k-2}\mathfrak{J}^{(1)}(\vec{u}) + \dots \tag{5.13}$$

where $\mathfrak{J}^{(n)}(\vec{u})$ is (up to additive constants and combinations of lower-level generators) the level-n generator with evaluation parameters $\vec{u}$. Now the preservation of Yangian invariance can be encoded in the RLL relation

$$R_{ij}(u_j - u_i)L_i(u_0 - \tfrac{1}{2}u_i)L_j(u_0 - \tfrac{1}{2}u_j) = L_i(u_0 - \tfrac{1}{2}u_j)L_j(u_0 - \tfrac{1}{2}u_i)R_{ij}(u_j - u_i)\,, \tag{5.14}$$

which again holds when the operators act in the space of Yangian invariant functions. Pictorially:



(5.15)

The R-operator moreover satisfies the Yang–Baxter equation (cf. the triangle equation above),[9]

$$R_{ij}(w - v)R_{j\ell}(w - u)R_{ij}(v - u) = R_{j\ell}(v - u)R_{ij}(w - u)R_{j\ell}(w - v)\,. \tag{5.16}$$

Taking all of this together, we have found an R-matrix for the representation of $\mathrm{Y}(\mathfrak{osp}(6|4))$ that can be written in Graßmanninan integral form, which seems to not have been known.

Invariants. We can now construct Yangian invariants by acting with a chain of R-matrices on a suitable "vacuum",

$$R_{i_\ell, j_\ell}(z_\ell)\dots R_{i_1, j_1}(z_1)\, \Omega_{2k}\,, \tag{5.17}$$

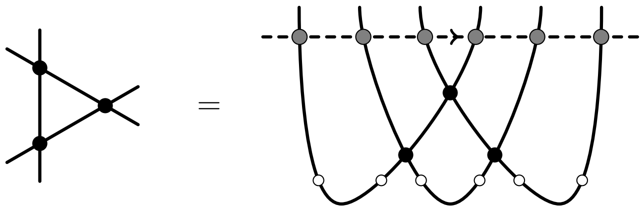
where the vacuum Ω_{2k} is given by the simplest possible $2k$-point invariant, which is a product of two-point invariants:

$$\Omega_{2k} = \prod_{j=1}^{k} \delta^{2|3}(\Lambda_{2j-1} + i\Lambda_{2j})\,. \tag{5.18}$$

[8] See e.g. [12].

[9] This can be shown to hold by noting that the rotations in the three R-matrices parametrize the three-dimensional rotation group in terms of Euler angles. The two sides of the equation are related by a coordinate transformation. One can verify explicitly that the product of measure factors in the integrals is kept invariant by the coordinate transformation.

The invariance of (5.17) can now be shown by acting with the monodromy operator. Using the RLL relation, the R-matrices can be pulled through the monodromy operator one by one, until the monodromy acts on the vacuum. One can easily see that the vacuum is an eigenstate of the monodromy, with eigenvalue $\prod_{j=1}^{2} k(u_0 - u_j/2)$, and hence (5.17) is annihilated by all Yangian generators. For example, the action of the monodromy on the simple six-point diagram (5.5) can be pictorially represented as

$T(u_0, \vec{u})$  (5.19)

The monodromy matrix (chain of Lax operators, gray dots) can be pulled through the R-matrices (black dots) until it acts on the vacuum (below the white dots).

6 Amplitudes and the Deformed Graßmannian Integral

Now that we have found Yangian-invariant deformations of all on-shell diagrams, can we use them to construct deformed invariant amplitudes (or loop integrands)? For MHV amplitudes in $\mathcal{N} = 4$ SYM (and for the four- and six-point amplitudes in ABJM theory), the tree amplitudes consist of a single on-shell diagram; in this case we already have a consistent deformation. What about higher-point and non-MHV amplitudes? Generally, tree amplitudes and loop integrands can be written as a sum of BCFW terms, and each term in the decomposition equals a single (undeformed) on-shell diagram. Can one combine the deformed diagrams in a similar way?

In order to sum multiple diagrams sensibly, they must all live in the same representation of the Yangian, that is the same evaluation parameters must be associated to their external legs. In the $\mathcal{N} = 4$ SYM case, we also want to demand that the central charges are the same on all diagrams in a sum.[10] In such a putative sum of m deformed diagrams, the evaluation parameters $u_{1\dots n}$ and central charges $c_{1\dots n}$ must obey the invariance conditions

$$u_i^+ = u_{\sigma(i)}^- \tag{6.1}$$

for all m permutations σ associated to the individual diagrams. (For ABJM, the c_i must be set to zero in these equations.) Experimentally, one finds that these relations do not leave room for non-trivial u_i and c_i beyond the six-point amplitude in $\mathcal{N} = 4$ SYM and the eight-point amplitude in ABJM theory.[11] The failure of a naive summation of deformed diagrams is no surprise, as the number of BCFW terms grows factorially, and each term requires n constraints on the deformation parameters.

Is there still a way to define a Yangian-invariant deformed amplitude, even though deforming the BCFW expansion term by term fails? A natural starting point to investigate this question is the Graßmannian integral. After all, this integral produces all the on-shell diagrams (BCFW terms).

[10] We could relax the condition of uniform central charges, but the interpretation of a sum of diagrams with non-uniform central charges would be unclear.

[11] The permutations for any tree amplitude are constructed in [5] for $\mathcal{N} = 4$ SYM, and in [10] for ABJM. For $\mathcal{N} = 4$ SYM, they are provided by the Mathematica package [13].

The Deformed Graßmannian Integral. In fact, the Graßmannian integral can be deformed directly. This can be seen immediately by noting that the Graßmannian integral is nothing but the top-cell diagram, that is the reduced diagram associated to the permutation $\sigma_{\text{top}} = (k+1, k+2, \dots, k)$, a k-fold cyclic shift. It therefore admits a deformation with $u_i^+ = u_{\sigma(i)}^-$. The deformed integral can be written as[12]

$$\mathcal{G}_{n,k}(\mathcal{W}_i) = \int \frac{\mathrm{d}^{k\cdot n}C}{|\mathrm{GL}(k)|} \frac{\delta^{4k|4k}(C\cdot\mathcal{W})}{M_1^{1+b_1}\cdot\ldots\cdot M_n^{1+b_n}}\,. \tag{6.2}$$

Since the twistors $\mathcal{W}_i$ only appear in the delta functions, the action of the local central charge operator

$$\mathfrak{C}_i = -\mathcal{W}_i^{\mathcal{C}}\frac{\partial}{\partial \mathcal{W}_i^{\mathcal{C}}} \tag{6.3}$$

can be transformed into a rescaling of column i of the matrix C. This allows to relate the exponents b_i to the central charges,

$$c_i = -(b_{i-k+1} + \cdots + b_i)\,, \tag{6.4}$$

which implies

$$b_i = \tfrac{1}{2}(u_i^- - u_{i-1}^-) = \tfrac{1}{2}(u_{i-k}^+ - u_{i-k-1}^+)\,. \tag{6.5}$$

At the level of the Graßmannian integral, we can therefore define a consistent deformation for all tree-level amplitudes!

The question is how this integral is to be interpreted. As in the undeformed case, the bosonic delta functions can be used to localize $(2n-4)$ of the integrations. But for non-MHV amplitudes, some integration variables remain. In the undeformed case, these are localized on poles at zeros of the minors M_i. But in the deformed case, the poles degenerate into cuts due to the non-integer exponents. One could reduce the space of deformation parameters by setting some of the b_i to zero, and localize the integration on the poles as in the undeformed case. Doing this in all possible ways reproduces all deformed on-shell diagrams that I discussed above. But for reproducing the tree amplitudes in the undeformed limit, one quickly runs into the case that *all* the parameters b_i need to be set to zero. This just corresponds to the fact that the BCFW decomposition cannot be deformed consistently term by term.

The task is thus to find a sensible contour for performing the remaining integrations in the Graßmannian integral in the presence of non-zero deformation parameters.

7 Outlook

We have significantly enlarged the class of known invariants of the Yangian representations relevant for scattering amplitudes, which can be seen as an interesting result in its own right. Whether these generalized invariants will be useful for scattering amplitudes is not entirely conclusive at present. But the fact that the complete Yangian symmetry can be preserved, and the similarity to known integrable structures gives reason for hope that they will be useful for the study of scattering amplitudes. Immediate tasks include:

- Find a useful integration contour.

[12] Here I will focus on the $\mathcal{N}=4$ SYM case. All the statements hold analogously for ABJM theory.

- Once a contour is specified, investigate deformed loop integrands. These are combinations of tree amplitudes and their forward limits; hence once deformed tree amplitudes are defined, the study of deformed loop integrands should be possible.
- "Lift" the deformations to the amplituhedron [14]. This is a geometric object that manifestly sums all BCFW terms, hence the problem of combining multiple deformed terms should not arise in the first place.

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References

[1] T. Bargheer, Y.-t. Huang, F. Loebbert and M. Yamazaki, *"Integrable Amplitude Deformations for $\mathcal{N}=4$ Super Yang–Mills and ABJM Theory"*, arxiv:1407.4449.

[2] L. Ferro, T. Łukowski and M. Staudacher, *"$\mathcal{N}=4$ scattering amplitudes and the deformed Graßmannian"*, Nucl. Phys. B889, 192 (2014), arxiv:1407.6736.

[3] L. Ferro, T. Łukowski, C. Meneghelli, J. Plefka and M. Staudacher, *"Harmonic R-matrices for Scattering Amplitudes and Spectral Regularization"*, Phys. Rev. Lett. 110, 121602 (2013), arxiv:1212.0850. • L. Ferro, T. Łukowski, C. Meneghelli, J. Plefka and M. Staudacher, *"Spectral Parameters for Scattering Amplitudes in $\mathcal{N}=4$ Super Yang–Mills Theory"*, JHEP 1401, 094 (2014), arxiv:1308.3494.

[4] N. Beisert, J. Broedel and M. Rosso, *"On Yangian-invariant regularization of deformed on-shell diagrams in $\mathcal{N}=4$ super-Yang–Mills theory"*, J. Phys. A47, 365402 (2014), arxiv:1401.7274.

[5] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, A. B. Goncharov, A. Postnikov and J. Trnka, *"Scattering Amplitudes and the Positive Grassmannian"*, arxiv:1212.5605.

[6] J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, *"Dual superconformal symmetry of scattering amplitudes in $\mathcal{N}=4$ super-Yang–Mills theory"*, Nucl. Phys. B828, 317 (2010), arxiv:0807.1095.

[7] J. M. Drummond, J. M. Henn and J. Plefka, *"Yangian symmetry of scattering amplitudes in $\mathcal{N}=4$ super Yang–Mills theory"*, JHEP 0905, 046 (2009), arxiv:0902.2987.

[8] B. I. Zwiebel, *"From Scattering Amplitudes to the Dilatation Generator in $\mathcal{N}=4$ SYM"*, J. Phys A45, 115401 (2012), arxiv:1111.0083.

[9] S. Lee, *"Yangian Invariant Scattering Amplitudes in Supersymmetric Chern–Simons Theory"*, Phys. Rev. Lett. 105, 151603 (2010), arxiv:1007.4772.

[10] Y.-T. Huang and C. Wen, *"ABJM amplitudes and the positive orthogonal grassmannian"*, JHEP 1402, 104 (2014), arxiv:1309.3252.

[11] Y.-t. Huang, C. Wen and D. Xie, *"The Positive orthogonal Grassmannian and loop amplitudes of ABJM"*, arxiv:1402.1479.

[12] L. Dolan and C. R. Nappi, *"Spin models and superconformal Yang–Mills theory"*, Nucl. Phys. B717, 361 (2005), hep-th/0411020.

[13] J. L. Bourjaily, *"Positroids, Plabic Graphs, and Scattering Amplitudes in Mathematica"*, arxiv:1212.6974.

[14] N. Arkani-Hamed and J. Trnka, *"The Amplituhedron"*, JHEP 1410, 30 (2014), arxiv:1312.2007. • N. Arkani-Hamed and J. Trnka, *"Into the Amplituhedron"*, arxiv:1312.7878.