Holographic Three-Point Functions in $\mathcal{N} = 4$ super Yang–Mills Theory

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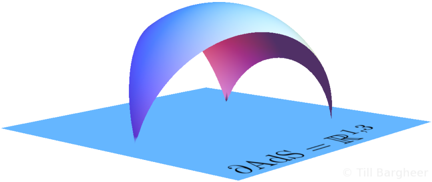

Based on work in progress with

Thomas Klose (UU), Tristan McLoughlin (AEI Potsdam), Anton Nedelin (UU)

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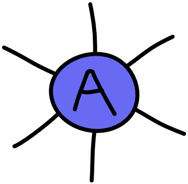
ETH Zürich

Why $\mathcal{N} = 4$ super Yang–Mills Theory?

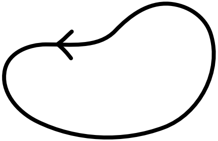
- Simplest interacting 4d QFT
- AdS/CFT and integrability
- Can solve for and compute many observables: Spectrum, S-matrix, null polygonal Wilson loops,...
- A lot of symmetry: Superconformal $\mathfrak{psu}(2,2|4)$, dual $\mathfrak{psu}(2,2|4)$, Yangian $Y(\mathfrak{psu}(2,2|4))$
- Most important properties:
  - UV finite
  - Planar limit: $N_\mathrm{c} \to \infty$, $\lambda \sim g_\mathrm{YM}^2/N_\mathrm{c}$ fixed
  - Relation to strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$ via AdS/CFT

Observables

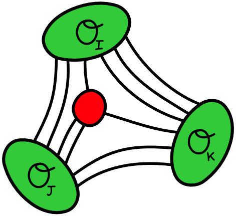
- Scattering amplitudes



- Wilson loops



- Correlation functions of local gauge-invariant operators



Correlation Functions in CFT

Correlation functions of scalar conformal primary operators largely fixed by conformal symmetry:

$$\langle \mathcal{O}_J(x_1)\mathcal{O}_K(x_2)\rangle = \frac{\delta_{JK}}{|x_{12}|^{\Delta_J+\Delta_K}}$$

$$\langle \mathcal{O}_I(x_1)\mathcal{O}_J(x_2)\mathcal{O}_K(x_3)\rangle = \frac{C_{IJK}}{|x_{12}|^{\Delta_I+\Delta_J-\Delta_K}|x_{23}|^{\Delta_J+\Delta_K-\Delta_I}|x_{31}|^{\Delta_K+\Delta_I-\Delta_J}}$$

$$\langle \mathcal{O}_I(x_1)\mathcal{O}_J(x_2)\mathcal{O}_K(x_3)\mathcal{O}_L(x_4)\rangle = C_{IJKL}\, f\left(\frac{x_{ij}x_{kl}}{x_{ik}x_{jl}}, \frac{x_{ij}x_{kl}}{x_{il}x_{jk}}\right)\prod_{i<j}\frac{1}{|x_{ij}|^{\Delta_i+\Delta_j-\Delta/3}}$$

$$\vdots$$

C_{IJKL} and f follow from $\{\Delta_I\}$, $\{C_{IJK}\}$ and the OPE.

Complete information encoded in $\{\Delta_J\}$ and $\{C_{IJK}\}$.

Two-Point Functions ↔ Scaling Dimensions

$\{\Delta_I\}$ in the planar limit from AdS/CFT and integrability [Beisert et al. 1012.3982 Lett.Math.Phys.99]

- Weak coupling, $\lambda \ll 1$:
  - Δ_I: Eigenvalues of dilatation generator
  - Spin-chain picture, exploit symmetries
  - Asymptotic all-loop Bethe equations
- Strong coupling, $\lambda \gg 1$:
  - Δ_I: Energies of physical strings
  - Worldsheet scattering of excitations
  - Asymptotic Bethe equations
- Exact, any λ:
  - Wrapping corrections
  - Thermodynamic Bethe ansatz
  - Y-system equations

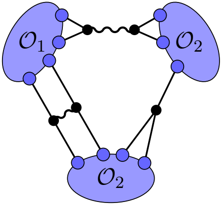
Upshot: No need to compute two-point functions directly!

Three-Point Functions ↔ Structure Constants

Goal (dream): C_{IJK} from integrability

But: No structure as for two-point functions known

⟶ For the moment, need to compute correlation functions directly.



Weak coupling:

Straightforward Feynman diagrams [Kristjansen, Plefka Semenoff, Staudacher '02] [Constable, Freedman Headrick, Minwalla '02] [Chu, Khoze Travaglini '02] [Beisert, Kristjansen, Plefka Semenoff, Staudacher '02]

More recently: Algebraic Bethe ansatz approach, label operators by Bethe roots, organize/simplify combinatorics

[Escobedo, Gromov Sever, Vieira 1012.2475] [Escobedo, Gromov Sever, Vieira 1104.5501] [Gromov Sever, Vieira 1111.2349] [Gromov Vieira 1202.4103]

Correlation Functions from Holography

Essence of AdS/CFT: [Maldacena hep-th/9711200] [Gubser, Klebanov, Polyakov hep-th/9802109] [Witten hep-th/9802150]

$$Z_{\text{string}}\left[\Phi\big|_{\partial\text{AdS}} = \Phi_0\right] = Z_{\text{CFT}}\left[\Phi_0\right]$$

More concretely, [Tseytlin hep-th/0304139] [Buchbinder, Tseytlin 1005.4516]

$$\left\langle e^{\Phi\cdot\mathcal{O}}\right\rangle_{\text{CFT}} = \left\langle e^{\Phi\cdot V}\right\rangle_{\text{string worldsheet}}$$

$$\Phi\cdot\mathcal{O}_{\text{CFT}} = \int \mathrm{d}^4x\,\Phi(x)\mathcal{O}(x)\,, \qquad \Phi\cdot V_{\text{string}} = \int \mathrm{d}^4x\,\Phi(x)V(x)\,.$$

Correlation Function:

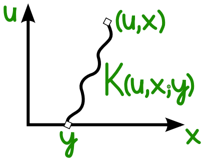
$$\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle = \int \mathcal{D}\mathbb{X}\,V_1V_2V_3\,e^{-S_{\text{string}}[\mathbb{X}]}$$

Dictionary $\mathcal{O}_{\text{CFT}} \leftrightarrow V_{\text{string}}$ not clear!

Supergravity Approximation

Simplest approximation: $\sqrt{\lambda} \to \infty$ $(\alpha' \to 0)$ and $m^2 \sim \Delta$ small
Massless string modes $\leftrightarrow$ chiral primary operators
Correlation functions protected by supersymmetry

Poincaré coordinates: $ds^2 = \dfrac{du^2 + dx^2}{u^2}$



Boundary field $\phi_0(x)$ [Freedman, Mathur Matusis, Rastelli '98]

Bulk sugra field $\phi(u,x) = \int \mathrm{d}^4 y \, K(u,x;y)\phi_0(y)$

Bulk-to-boundary propagator $K(u,x;y) = \left(\dfrac{u}{u^2 + (x-y)^2}\right)^{\Delta}$

AdS/CFT: $Z_{\mathrm{CFT}}[\phi_0] = Z_{\mathrm{string}}[\phi] = \left\langle e^{-S_{\mathrm{sugra}}[\phi]} \right\rangle$

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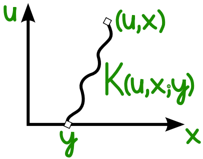

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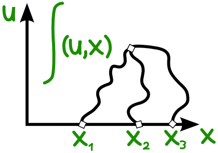
AdS/CFT: $Z_{\mathrm{CFT}}[\phi_0] = Z_{\mathrm{string}}[\phi] = \left\langle e^{-S_{\mathrm{sugra}}[\phi]} \right\rangle$

CFT: Chiral primary $\mathcal{S}_I = C_I^{j_1 \ldots j_k} \, \mathrm{Tr}(\phi_{j_1} \ldots \phi_{j_k})$

Dual sugra field contains $h_{\alpha\beta}$, $a_{\alpha\beta\gamma\delta}$



For all λ: [Lee, Minwalla Rangamani, Seiberg '98]

$$\left\langle \mathcal{S}_I(x_1)\mathcal{S}_J(x_2)\mathcal{S}_K(x_3) \right\rangle = \frac{1}{N_c} \frac{\sqrt{\Delta_1\Delta_2\Delta_3}\langle C_I C_J C_K \rangle}{|x_{12}|^{\Delta_I+\Delta_J-\Delta_K}|x_{23}|^{\Delta_J+\Delta_K-\Delta_I}|x_{31}|^{\Delta_K+\Delta_I-\Delta_J}}$$

Semiclassical Strings

Large Noether charges: $\varDelta, J, S \sim \sqrt{\lambda} \to \infty$
Classical strings good approximation, quantum fluctuations small
Access to massive string modes $\leftrightarrow$ unprotected operators
Successfully employed in spectral problem

- Classical non-linear sigma model is integrable [Bena, Polchinski Roiban, hep-th/0305116]
- Scattering of quantum fluctuations on world-sheet
- Description in terms of integrable spin-chain

Examples of semiclassical string states:

- BMN: Point-like string orbiting $\mathrm{S}^5 \leftrightarrow \operatorname{Tr} Z^J$ [Berenstein, Maldacena Nastase hep-th/0202021]
- GKP: Folded string spinning in $\mathrm{AdS}_5 \leftrightarrow \operatorname{Tr} Z\mathcal{D}^S Z$ [Gubser, Klebanov Polyakov, hep-th/0204051]
- Circular spinning string: Spinning in AdS and on sphere [Frolov, Tseytlin hep-th/0304255]

Path Integral and Vertex Operators I

For $\sqrt{\lambda} \to \infty$, path integral

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = \int \mathcal{D}\mathbb{X}\, V_1 V_2 V_3\, e^{-\sqrt{\lambda} \int \mathrm{d}^2 z\, \mathcal{L}_{\text{string}}[\mathbb{X}]}$$

dominated by saddle point $\rightarrow$ classical solution

For large charges $\Delta, J, S \sim \sqrt{\lambda}$, vertex operators of same order as e^{-S}

General structure: $V \sim e^{i\,\text{Charge}\cdot\text{Coordinate}}$ (Polynomial)

Flat-space example: $V \sim e^{ik_\mu x^\mu}$ sources tachyon modes

AdS: Poincaré u, x Embedding Y Global $t, \phi, \psi, \ldots$

$$ds^2 = \frac{du^2 + dx^2}{u^2} \qquad ds^2 = \sum_k \pm dY_k^2$$

Path Integral and Vertex Operators II

Translating $V \sim e^{i\Delta t}$ to Poincaré/embedding coordinates, it becomes

$$V(z;x_k) \sim \left(\frac{u(z)}{u(z)^2 + \left(x(z) - x_k\right)^2}\right)^{\Delta} \sim (Y_{+,k})^{-\Delta}$$

Exactly the sugra bulk-to-boundary propagator!
Varying the path integral

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = \int \mathcal{D}\mathbb{X}\, V_1(z_1;x_1)\, V_2(z_2;x_2)\, V_3(z_3;x_3)\, e^{-\sqrt{\lambda}\int \mathrm{d}^2 z\, \mathcal{L}_{\text{string}}}\,,$$

for $\Delta_k \sim \sqrt{\lambda}$, the vertex operators V_k induce source terms in the classical string equations of motion:

$$\text{EOM}_{\text{string}} \quad \longrightarrow \quad \text{EOM}_{\text{string}} + \sum_{k=1}^{3} \frac{\Delta_k}{\sqrt{\lambda}}\, \delta^2(z - z_k) F_k(z)$$

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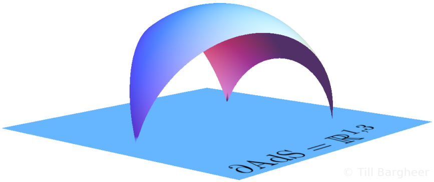
At the AdS boundary: $(Y_{+,k})^{-1} \xrightarrow{u \to 0} \delta^4(x - x_k)$
Ensures that classical solution ends on points x_k on $\partial\mathrm{AdS}$



Vertex Operators: Examples

General form in semiclassical limit (U polynomial):

$$V = (Y_+)^{-\Delta}\, U(\mathbb{X}, \partial\mathbb{X}, \dots)\,, \qquad \mathbb{X} = (Y, X)$$

No complete dictionary for vertex operators
Constraints:

- V has to be marginal perturbation of string sigma model
- V has to reproduce right charges

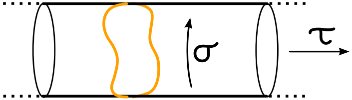
Some examples: [Polyakov hep-th/0110196] [Tseytlin hep-th/0304139] [Buchbinder, Tseytlin 1005.4516]

- BMN (point-like string): $V_J = (Y_+)^{-\Delta}(X_1 + iX_2)^J$
- GKP (folded spinning string): $V_S = (Y_+)^{-\Delta}(\partial Y \bar{\partial} Y)^{S/2}$
- Spinning string: E.g. $V_J = (Y_+)^{-\Delta}(\partial X \bar{\partial} X)^{J/2}$

V_k determine asymptotics of solution near operator insertion points x_k

[Polyakov 0110196] [Tseytlin hep-th/0304139] Two-Point Functions [Buchbinder 1002.1716] [Buchbinder, Tseytlin 1005.4516]

Take a classical string solution on Minkowski cylinder (τ, σ)



Apply Wick rotation on worldsheet $\tau \to i\tau$ and in target space $t \to it$

Map cylinder (τ, σ) to complex plane z by

$$e^{\tau + i\sigma} = \frac{z - z_1}{z - z_2}$$

Asymptotic points $\tau = \pm\infty$ are mapped to punctures at $z_{1,2}$

Resulting configuration is semiclassical saddle point

- Satisfies string EOM with vertex operator insertions at $z_{1,2}$
- Vertex operators carry same charges as initial string solution

Heavy-Heavy-Light

Have simplified the problem to finding solution to classical string EOM with prescribed sources (from vertex operators)

Still a difficult problem!

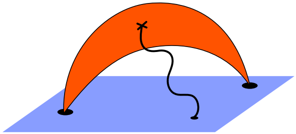
Further simplification:

- Two asymptotic heavy states $\Delta \sim \sqrt{\lambda}$
- One massless/light state, $\Delta \sim 1$ or $\Delta \sim \sqrt[4]{\lambda}$ (sugra mode)

$\Longrightarrow$ Hybrid formulation of string theory and supergravity [Tseytlin '90 Phys.Lett.B251]

Path integral dominated by classical two-point solution

Leading contribution to correlator:
Supergravity mode insertion integrated over worldsheet, connected to boundary via bulk-to-boundary propagator



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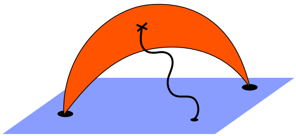


Results:

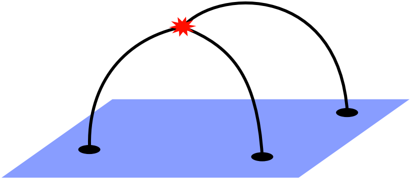
- Folded spinning string + CPO dual [Zarembo 1008.1059]
- Various strings + dilaton. Also gauge theory. [Costa, Monteiro Santos, Zoakos, 1008.1070]

Activity: Spinning strings, folded strings, giant magnon, giant graviton, finite size . . .
[Roiban, Tseytlin 1008.4921] [Hernandez 1011.0408] [Ryang 1011.3573] [Georgiou 1011.5181] [Park, Lee 1012.3293] [Buchbinder, Tseytlin 1012.3740] [Bissi, Kristjansen, Young Zoubos 1103.4079] [Arnaudov, Rashkov Vetsov 1103.6145] [Hernandez 1104.1160] [Bai, Lee, Park 1104.1896] [Ahn, Bozhilov 1105.3084] [Lee, Park 1105.3279] [more . . .]

AdS Geodesics

Simplest case beyond heavy-heavy-light:
Large spins only on the sphere, $J \sim \Delta$

- Asymptotic states are point-like geodesics in AdS
- Three geodesics glued together at a point [Klose, McLoughlin 1106.0495]
- Saddle-point found by extremizing action w.r.t. interaction point
- Contribution of the sphere also included



Wanted: Heavy-Heavy-Heavy

Correlator of three states with large AdS charges
Qualitatively different from

- Three geodesics: Non-trivial worldsheet
- Heavy-heavy-light: Worldsheet is not a cylinder

Want: Solution to classical string EOM with

- Vertex operator sources
- Prescribed two-point asymptotics at three points

AdS/$\mathbb{H}$: $0 = \partial\bar{\partial}Y - (\partial Y \cdot \bar{\partial}Y)Y$

Sphere: $0 = \partial\bar{\partial}X + (\partial X \cdot \bar{\partial}X)X$

Virasoro: $(\partial Y)^2 + (\partial X)^2 = 0\,, \qquad (\bar{\partial}Y)^2 + (\bar{\partial}X)^2 = 0$

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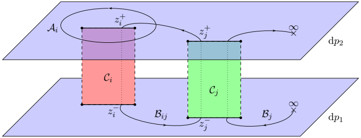
Use Integrability?

Find explicit solution and evaluate action. Looks difficult!

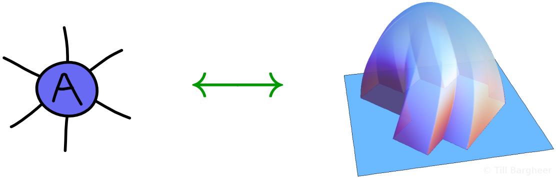
But non-linear sigma model on symmetric space is integrable [Bena, Polchinski Roiban, 0305116]

Has been very successfully exploited in

- Spectral problem: Algebraic curve



- Scattering amplitudes: Find surface, evaluate action [Alday, Maldacena 0904.0663]



Can we use integrability to find a solution / evaluate the action?

Split AdS/Sphere

Saddle point of the path integral factorizes:

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle \approx V_1\, V_2\, V_3\, e^{-S}\Big|_{\text{saddle point}} = G_{\text{AdS}}\, G_{\text{Sphere}}$$

Concentrate on AdS part

Virasoro constraints link AdS and sphere:

$$(\partial Y)^2 + (\partial X)^2 = 0\,, \qquad (\bar{\partial} Y)^2 + (\bar{\partial} X)^2 = 0$$

Rewrite as

$$\partial Y(z) \cdot \partial Y(z) = -T(z)\,, \qquad \bar{\partial} Y(z) \cdot \bar{\partial} Y(z) = -\bar{T}(z)$$

with "energy-momentum tensor" T, $\bar{T}$.

The Energy-Momentum Tensor

T, $\bar{T}$ a priori depend on the dynamics on the sphere

$$T(z) = -\partial Y \cdot \partial Y = +\partial X \cdot \partial X$$
$$\bar{T}(z) = -\bar{\partial} Y \cdot \bar{\partial} Y = +\bar{\partial} X \cdot \bar{\partial} X$$

But can be constrained by [Janik, Wereszczyński 1109.6262]

- Two-point asymptotics at the insertion points
- Transformation under conformal inversions:

$$T(z; z_k) = z^4\, T(1/z; 1/z_k)$$

For three states without large AdS spins, T completely fixed

$$T(z) \xrightarrow{z \to z_k} \frac{\Delta_k^2}{4\lambda\,(z - z_k^2)}\,, \qquad T(z) = \frac{P(z)}{(z-z_1)^2\,(z-z_2)^2\,(z-z_3)^2}$$

Reasonable to assume that this is true also for other states

Pohlmeyer Reduction

Reduction of EOM to generalized complex sinh-Gordon:

AdS_3: Basis of embedding space $\{Y, \partial Y, \bar{\partial} Y, N\}$

$$\cosh\alpha := \frac{\partial\vec{Y}\cdot\bar{\partial}\vec{Y}}{\sqrt{T\bar{T}}}\,, \qquad p := \frac{\vec{N}\cdot\partial^2\vec{Y}}{T}\,, \qquad \bar{p} := \frac{\vec{N}\cdot\bar{\partial}^2\vec{Y}}{\bar{T}}$$

Take derivatives, apply EOM for Y, expand in basis:

$$\partial\bar{\partial}\alpha = \sqrt{T\bar{T}}\left(\sinh\alpha + \frac{p\bar{p}}{\sinh\alpha}\right)\,, \quad \bar{\partial}p = -\frac{\sqrt{\bar{T}}\,\bar{p}\,\partial\alpha}{\sqrt{T}\sinh\alpha}\,, \quad \partial\bar{p} = -\frac{\sqrt{T}\,p\,\bar{\partial}\alpha}{\sqrt{\bar{T}}\sinh\alpha}$$

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Integrable system!

Known solutions and techniques

But: Surface only implicit, reconstruction difficult

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Reconstruction of the Surface

Take basis $q_\mu = \{Y, \partial Y, \bar\partial Y, N\}$ and expand derivatives

$$\partial q_\mu = q_\nu \, B^\nu{}_\mu \,, \qquad \bar\partial q_\mu = q_\nu \, \bar B^\nu{}_\mu \qquad \text{(Gauss-Weingarten)}$$

where $B = B(\sqrt{T}, p, \alpha, \partial\alpha)$ and $\bar B$ are 4×4 matrices.

Write q in spinor indices, and split

$$q^m_\mu = q_{\alpha\dot\alpha, a\dot a} \, \sigma^{\alpha\dot\alpha}_\mu \, \sigma^{m, a\dot a} \,, \qquad q_{\alpha\dot\alpha, a\dot a} = \psi^{\mathrm{L}}_{\alpha, a} \psi^{\mathrm{R}}_{\dot\alpha, \dot a}$$

Left and right 2×2 auxiliary linear problem

$$\partial\psi^{\mathrm{L}}_{\alpha,a} + (B^{\mathrm{L}}_z)_\alpha{}^\beta \psi^{\mathrm{L}}_{\beta,a} = 0 \,, \qquad \partial\psi^{\mathrm{R}}_{\dot\alpha,\dot a} + (B^{\mathrm{R}}_z)_{\dot\alpha}{}^{\dot\beta} \psi^{\mathrm{R}}_{\dot\beta,\dot a} = 0 \,,$$
$$\bar\partial\psi^{\mathrm{L}}_{\alpha,a} + (B^{\mathrm{L}}_{\bar z})_\alpha{}^\beta \psi^{\mathrm{L}}_{\beta,a} = 0 \,, \qquad \bar\partial\psi^{\mathrm{R}}_{\dot\alpha,\dot a} + (B^{\mathrm{R}}_{\bar z})_{\dot\alpha}{}^{\dot\beta} \psi^{\mathrm{R}}_{\dot\beta,\dot a} = 0 \,.$$

Compatibility condition: Connections $B^{\mathrm{L,R}}$ are flat

$$\partial \bar B^{\mathrm{L,R}} - \bar\partial B^{\mathrm{L,R}} + [B^{\mathrm{L,R}}, \bar B^{\mathrm{L,R}}] = 0 \,. \qquad \text{(Gauss-Codazzi)}$$

Reconstruction of the Surface

Take basis $q_\mu = \{Y, \partial Y, \bar\partial Y, N\}$ and expand derivatives

$$\partial q_\mu = q_\nu\, B^\nu{}_\mu\,, \qquad \bar\partial q_\mu = q_\nu\, \bar B^\nu{}_\mu \qquad \text{(Gauss-Weingarten)}$$

where $B = B(\sqrt{T}, p, \alpha, \partial\alpha)$ and $\bar B$ are 4×4 matrices.

Write q in spinor indices, and split

$$q_\mu^m = q_{\alpha\dot\alpha, a\dot a}\, \sigma_\mu^{\alpha\dot\alpha}\, \sigma^{m,a\dot a}\,, \qquad q_{\alpha\dot\alpha,a\dot a} = \psi^{\mathrm{L}}_{\alpha,a}\psi^{\mathrm{R}}_{\dot\alpha,\dot a}$$

Left and right 2×2 auxiliary linear problem

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$$\begin{aligned} \partial\psi^{\mathrm{L}}_{\alpha,a} + (B^{\mathrm{L}}_z)_\alpha{}^\beta \psi^{\mathrm{L}}_{\beta,a} &= 0 \,, & \partial\psi^{\mathrm{R}}_{\dot\alpha,\dot a} + (B^{\mathrm{R}}_z)_{\dot\alpha}{}^{\dot\beta} \psi^{\mathrm{R}}_{\dot\beta,\dot a} &= 0 \,, \\ \bar\partial\psi^{\mathrm{L}}_{\alpha,a} + (B^{\mathrm{L}}_{\bar z})_\alpha{}^\beta \psi^{\mathrm{L}}_{\beta,a} &= 0 \,, & \bar\partial\psi^{\mathrm{R}}_{\dot\alpha,\dot a} + (B^{\mathrm{R}}_{\bar z})_{\dot\alpha}{}^{\dot\beta} \psi^{\mathrm{R}}_{\dot\beta,\dot a} &= 0 \,. \end{aligned}$$

Compatibility condition: Connections $B^{\mathrm{L,R}}$ are flat

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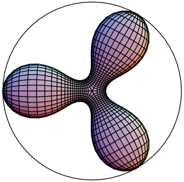
Explicit Construction?

Have split the problem into two pieces:

- Generalized sinh-Gordon equations $\partial\bar{\partial}\alpha = \ldots\,, \;\; \partial\bar{p} = \ldots\,,$
- Auxiliary problem $\partial\psi + B\psi = 0$

Still difficult to solve

One approach: **Find explicit solution**
"N-point-like" surfaces in $\mathbb{H}_3$ (N-noids) known [Bobenko, Pavlyukevich Springborn, math/0206021]



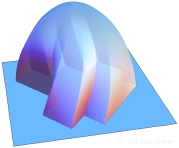
But: Not exactly right (no T, $\bar{T}$)
$\longrightarrow$ No luck so far.

Saddle Point from Asymptotics

Try to compute similar to Alday & Maldacena:

[Alday, Maldacena 0904.0663]

- Find saddle-point action without knowing the exact solution
- Exploit the integrability



Partially done for three folded spinning (GKP) strings

[Kazama, Komatsu 1110.3949]

And for BMN-like strings (geodesics)

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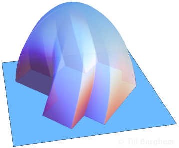

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Summary and Outlook

We are still at the beginning

Classical strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$ integrable
Adapt amplitude methods for semiclassical strings
Qualitative differences:

- Multiple asymptotic regions
- Logarithmic branch cuts

Can saddle-point action be recovered from asymptotic data?
Regularization/cancellation of divergences with vertex operators?
What about the sphere contribution?

Description of asymptotic states in terms of algebraic curves useful?