Two-Cut Solutions of the Heisenberg Ferromagnet and Stability

Till Bargheer

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Work with Niklas Beisert (to appear)
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Motivation

The Landau-Lifshitz Model

The Heisenberg Spin Chain and the Thermodynamic Limit

A Single Cut

Cuts Interact: The Two-Cut Solution
Heisenberg Spin Chain, Landau-Lifshitz model, Integrability

**LL model:** Limit of ultrarelativistic rotating string on $\mathbb{R} \times S^3$. [Kruczenski hep-th/0311203]

- Subspace of $\text{AdS}_5 \times S^5 \rightarrow \text{AdS/CFT}$ correspondence.
- Quantization around certain solutions $\rightarrow$ Unstable excitation modes.

**Heisenberg Spin Chain:** Quantum-mechanical model for 1D magnet

- Equivalent to $\text{SU}(2)$ sector of planar $\mathcal{N} = 4$ SYM (one-loop). [Minahan Zarembo]
- Solved (in principle) by the Bethe equations.
- Matches $\mathbb{R} \times S^3$ sector of $\text{AdS}_5 \times S^5$ string theory.
- Thermodynamic limit ($\infty$ long chain): Equivalent to the LL model $\rightarrow$ Heisenberg Chain is quantized version of LL model. [Kruczenski hep-th/0311203]

**Integrability:**

- LL model: Classical integrability $\rightarrow$ *Spectral curves*.
- Spin Chain: Quantum integrability $\rightarrow$ *Bethe equations*.
Aim of the Project

Goal: Find the spectrum of the Heisenberg ferromagnet in the thermodynamic limit.

- Understand the phase space of solutions to the Bethe equations.
- In particular: Investigate unstable modes of the LL model.
- Measurements on 1D-magnets?!
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The Landau-Lifshitz Model

- Classical non-relativistic sigma model on the sphere $S^2$ with fields $\phi$, $\theta$ and Lagrange function

$$L[\phi, \dot{\phi}, \theta] = -\frac{L}{4\pi} \int \cos \theta \dot{\phi} \, d\sigma - \frac{\pi}{2L} \int (\theta''^2 + \sin^2 \theta \phi''^2) \, d\sigma$$

- Effective model for the exact description of strings on $\mathbb{R} \times S^3$ that rotate at highly relativistic speed.

- Equations of motion:

$$\dot{\phi} = (2\pi/L)^2 \left( \cos(\theta)\phi''^2 - \csc(\theta)\theta'' \right),$$
$$\dot{\theta} = (2\pi/L)^2 \left( 2\cos(\theta)\theta' \phi' + \sin(\theta)\phi'' \right).$$

- Charges: Momentum $P$, spin $\alpha$, energy $\tilde{E}$:

$$P = \frac{1}{2} \int (1 - \cos(\theta)) \phi' \, d\sigma, \quad \alpha = \frac{1}{4\pi} \int (1 - \cos(\theta)) \, d\sigma,$$
$$\tilde{E} = \frac{\pi}{2} \int (\theta'^2 + \sin^2(\theta)\phi'^2) \, d\sigma.$$
Vacuum: Energy minimized by string localized at a point:

\[ \theta(\sigma, \tau) \equiv 0, \quad P = \alpha = \tilde{E} = 0. \]

A simple solution: Constant latitude \( \theta_0 \), \( n \) windings:

\[ \theta(\sigma, \tau) \equiv \theta_0, \quad \phi(\sigma, \tau) = n\sigma + (2\pi/L)^2 n^2 \cos(\theta_0)\tau. \]

Momentum \( P \) and energy \( \tilde{E} \) can be expressed in terms of spin \( \alpha \) and mode number \( n \):

\[ \alpha = \frac{1}{2}(1 - \cos(\theta_0)), \quad P = 2\pi n\alpha, \quad \tilde{E} = 4\pi^2 n^2 \alpha (1 - \alpha). \]

Semiclassical quantization \( \rightarrow \) Contact with Heisenberg model.
Fluctuation Modes

- Add excitations with mode number $k$ to the simple solution:

$$
\begin{align*}
\theta(\sigma, \tau) &= \theta_0 + \varepsilon \theta_+(\tau) e^{ik\sigma} + \varepsilon \theta_-(\tau) e^{-ik\sigma} + \varepsilon^2 \theta_c(\tau), \\
\phi(\sigma, \tau) &= \phi_0 + \varepsilon \phi_+(\tau) e^{ik\sigma} + \varepsilon \phi_-(\tau) e^{-ik\sigma} + \varepsilon^2 \phi_c(\tau).
\end{align*}
$$

- Expanding $L[\phi, \dot{\phi}, \theta]$ to order $\varepsilon^2$ yields two coupled HO’s with charges

$$
\begin{align*}
\delta \alpha &= 1/L, \\
\delta P &= 2\pi(n + k)/L, \\
\delta \tilde{E} &= (2\pi)^2/L \left( n(n + 2k)(1 - 2\alpha) + k^2 \sqrt{1 - 4n^2\alpha(1 - \alpha)/k^2} \right).
\end{align*}
$$

$\Rightarrow$ The classical solution becomes unstable when $2n\sqrt{\alpha(1 - \alpha)} > k = 1$, that is for large $n$ and $\alpha$. 
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The Heisenberg Spin Chain

- One of the oldest quantum mechanical models, set up by Heisenberg in 1928, describes a 1D-magnet with nearest-neighbor interaction of \( L \) spin-1/2 particles.

- Describes SU(2)-sector of planar \(\mathcal{N} = 4\) SYM at one loop.

- Hilbert space \(\mathcal{H}\) is the tensor product of \( L \) single-spin spaces \(\mathbb{C}^2\):
  \[
  \mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 = (\mathbb{C}^2)^\otimes L, \quad \text{e.g. } |\downarrow\downarrow\uparrow\downarrow\downarrow\uparrow\downarrow\downarrow\uparrow\rangle \in \mathcal{H}.
  \]

- The Hamiltonian \( H : \mathcal{H} \to \mathcal{H} \) is periodic
  \[
  H = \frac{1}{2} \sum_{k=1}^{L} (1 - \sigma_k \cdot \sigma_{k+1}) = \sum_{k=1}^{L} (1 - P_{k,k+1}).
  \]

- The energy spectrum is bounded between
  - The ferromagnetic ground state \( |\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\rangle\), energy \( E = 0 \).
  - The antiferromagnetic ground state “\( |\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\rangle\)”, \( E \approx L \log 4 \).
Bethe Equations

The Heisenberg spin chain is exactly solvable:

- Fundamental excitations: *Magnons* with definite momentum $p$ and rapidity $u = \frac{1}{2} \cot(p/2)$.
- All eigenstates can be explicitly constructed as combinations of multiple magnons (Bethe ansatz).
- Only requirement: Constituent magnons $u_1, \ldots, u_M$ must satisfy **Bethe equations** (ensure periodicity of wave function):

  \[
  \left( \frac{u_k + i/2}{u_k - i/2} \right)^L = \prod_{j=1}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}, \quad k = 1, \ldots, M.
  \]

- Momentum and Energy:

  \[
  e^{iP} = \prod_{k=1}^{M} \frac{u_k + i/2}{u_k - i/2}, \quad E = \sum_{k=1}^{M} \frac{1}{u_k^2 - 1/4}.
  \]
The Thermodynamic Limit

- Bethe equations hard to solve for more than a few excitations $u_k$.

- Problem simplifies in **thermodynamic limit**:  
  - Length of the chain (number of sites) $L \to \infty$.
  - Number of excitations (flipped spins) $M \to \infty$.
  - **Filling fraction** $\alpha = M/L$ fixed.
  - Keep only low-energy excitations (IR modes, coherent states), energies $E = \tilde{E}/L \sim 1/L$.

- In this limit, contact with the LL model is established: In coherent states, the expectation values of single spins form paths on $S^2$:

$\Rightarrow$ Classical LL model on $S^2$ is an effective model for the Heisenberg chain in the ferromagnetic thermodynamic limit.
In the thermodynamic limit, rescaled roots $x_k = u_k / L$ of coherent states condense on contours in the complex plane:

In the strict limit $L \to \infty$, the Bethe equations turn into integral equations which describe the positions of the contours and the root density along them. The contours constitute the branch cuts of the spectral curve, which is the solution of the LL model.
**Small Filling**

- **General picture for small fillings** $\alpha_i$:

  Positions of cuts: Integer mode numbers $n_i, x_k \sim 1/n_i$.
  Small densities, weak interaction between individual cuts.

- **When the filling of a cut grows**, the cut gets longer and its density increases.

- **Questions**:
  - What happens when root density approaches $|\rho| = 1$?
  - Singularity in the Bethe equations; construction of the spectral curve no longer valid!?
  - Relation to unstable modes in LL model?
  - Do corresponding solutions to the Bethe equations exist for all spectral curves?
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From Small to Large Filling

First: Study the configuration with a single cut in detail.

- As the cut grows, it attracts the neighboring fluctuation points:

  ![Diagram showing fluctuation points and cuts]

- Expect something interesting when

  - Fluctuation point collides with cut: Density reaches $|\rho| = 1$, Bethe equations singular.
  - Two successive fluctuation points collide and diverge into the complex plane. Instability? Coincides with filling where instability appears in the classical LL model.
Bethe Strings

- For the a priori infinite chain, individual magnon rapidities are arbitrary (no periodicity $\rightarrow$ no Bethe equations), but generically must be real.

- Only exception: “Bethe strings”. Bound states on the infinitely long chain, complex rapidities with regular pattern.

- Guess/Expect: Bethe strings appear when density on contour reaches $|\rho| = 1$. 

\[
\begin{array}{c}
0 & 1 & 2 & 3 & 4 \\
\hline \\
-4 & -2 & 0 & 2 & 4 \\
\end{array}
\]
Spectral curve that encodes the contours is a function with genus $g = 1$ (for two cuts). Find analytic expression.

Generalize the symmetric solution

Map two symmetric cuts to two general cuts via Möbius transformation $\mu$:

The parameters of $\mu$ and the position of the symmetric cuts leave enough freedom for placing the cuts at will.

Need to solve for specific solutions that obey the integral equations. Possible numerically.
Collision of Two Cuts

When the filling grows, cuts can collide and form condensates with density $|\rho| = \Delta u = 1$:

Cuts can even pass through each other:

The passing cut changes its mode number: $n_2 \rightarrow 2n_1 - n_2$. 
Closed Loop Cuts

Consider a very small second cut:

\[
\begin{array}{cccccc}
\times & \rightarrow & \times & \rightarrow & \times & \rightarrow \\
\times & \rightarrow & \times & \rightarrow & \times & \rightarrow \\
\end{array}
\]

Compare this to a bare excitation point that passes through:

\[
\begin{array}{cccccc}
\ast & \rightarrow & \ast & \rightarrow & \times & \rightarrow \\
\ast & \rightarrow & \ast & \rightarrow & \times & \rightarrow \\
\end{array}
\]

A closed loop with a condensate appears naturally.
Collision of Fluctuation Points: Instability

Look at the point where unstable modes in the LL model appear:

Consider again small cuts instead of bare excitation points:

Implication for closed cut (one-cut solution):
Phase transition

- Excitation of a mode means: Regular point $\rightarrow$ Two branch points.
- Fluctuation point real: Excitation $=$ Addition of roots.
- Fluctuation points complex with loop cut: Excitation means taking roots away.
  $\Rightarrow$ Behind instability point: Single cut $+$ loop $\rightarrow$ two cuts:
Further Features of Solutions

What if the fluctuation point is already excited at the instability point?

BPs can cycle around each other until they reach the imaginary line, $P = \pi$:
Phase Space of Consecutive Mode Numbers
Summary and Conclusions

- TD limit of Heisenberg Ferromagnet is equivalent to LL model.
- Macroscopic excitations are contours in complex plane.
- Apparent singularity of the Bethe equations in the TD limit is always hidden in a condensate.
- Moduli space is connected: Mode numbers can change.
- Phase space is “filled”: For all values of the moduli there are corresponding solutions to the Bethe equations.
- Unstable classical solutions are degenerate three-cut solutions that “decay” into two-cut solutions.
Numerical Results

Numerical verification:

\[ \begin{align*}
TB, Beisert, Gromov & \quad \text{to appear} \\
\end{align*} \]