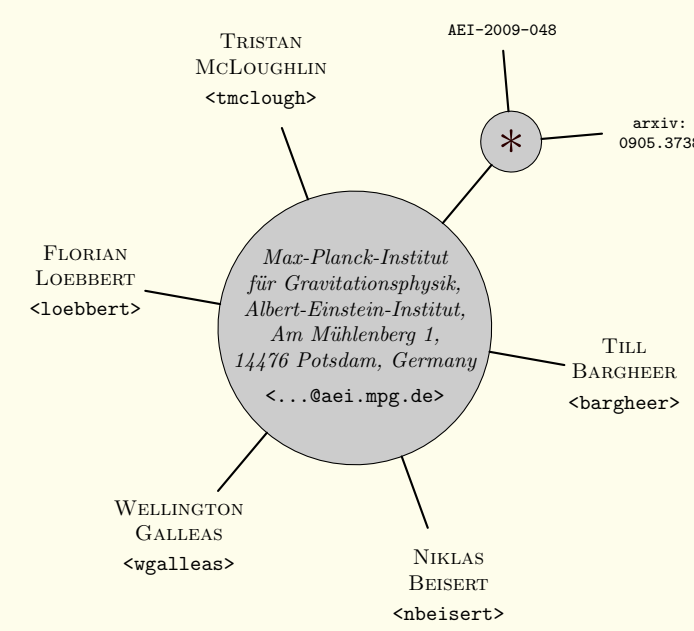




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Exacting $\mathcal{N} = 4$ Superconformal Symmetry*

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Scattering Amplitudes

Scattering amplitudes in $\mathcal{N} = 4$ SYM are conveniently expressed in the spinor helicity superspace: The light-like momentum p of each external particle is converted to a bi-spinor $p^{a\dot{a}} = \lambda^a \bar{\lambda}^{\dot{a}}$, where λ^a and $\bar{\lambda}^{\dot{a}}$ are conjugate bosonic Lorentz spinors with $a, \dot{a} = 1, 2$. It is advantageous to compute scattering amplitudes for the superfield $\Phi(\lambda, \bar{\lambda}, \eta)$ with η^A , $A = 1, \dots, 4$ being fermionic spinors. The scattering amplitude for n external particles is thus a superspace function

$$A_n(\lambda_1, \bar{\lambda}_1, \eta_1, \dots, \lambda_n, \bar{\lambda}_n, \eta_n) = \int \mathcal{A}$$

Amplitudes can be classified through their helicity measured by the number of η 's ranging between 8 for MHV and $4n - 8$ for $\overline{\text{MHV}}$ amplitudes. Tree level MHV amplitudes of $\mathcal{N} = 4$ SYM take the simple form [1]

$$A_n^{\text{MHV}} = \frac{\delta^4(P) \delta^8(Q)}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle},$$

where $\langle j, k \rangle = \varepsilon_{ab} \lambda_j^a \lambda_k^b$ and

$$P^{ab} = \sum_{k=1}^n \lambda_k^a \bar{\lambda}_k^b, \quad Q^{aB} = \sum_{k=1}^n \lambda_k^a \eta_k^B.$$

Free Representation

The free representation of the superconformal algebra on tree-level scattering amplitudes in $\mathcal{N} = 4$ SYM can be written in a compact fashion. Most relevantly

$$\begin{aligned} (\mathcal{Q}_0)^{aB} &= \lambda^a \eta^B, & (\mathcal{S}_0)_{aB} &= \partial_a \partial_B, \\ (\bar{\mathcal{Q}}_0)_{\dot{a}B} &= \bar{\lambda}^{\dot{a}} \partial_B, & (\bar{\mathcal{S}}_0)_{\dot{a}B} &= \eta^B \bar{\partial}_{\dot{a}}, \\ (\mathfrak{P}_0)^{ab} &= \lambda^a \bar{\lambda}^b, & (\mathfrak{R}_0)_{ab} &= \partial_a \bar{\partial}_b, \end{aligned} \quad (1)$$

where $\partial_a = \partial/\partial\lambda^a$, $\bar{\partial}_{\dot{a}} = \partial/\partial\bar{\lambda}_{\dot{a}}$ and $\partial_A = \partial/\partial\eta^A$. The action of the generators \mathfrak{J}_0 takes the standard tensor product form

$$\mathfrak{J}_0 = \sum_{k=1}^n \mathfrak{J}_{0,k} = \int \mathcal{A}$$

Here $\mathfrak{J}_{0,k}$ is the representation of the conformal symmetry generator \mathfrak{J}_0 on the k -th leg $(\lambda_k, \bar{\lambda}_k, \eta_k)$ of A_n as specified in (1). Invariance of A_n corresponds to the statement

$$\mathfrak{J}_0 A_n = 0. \quad (2)$$

Higher Loop Representation

Inspired by the representation of the symmetry generators on local operators [3], assume the following perturbative representation $\mathfrak{J}(g)$ for some symmetry generator \mathfrak{J} around the free representation $\mathfrak{J}_0 = \mathfrak{J}_{1,1}^{(0)}$:

$$\begin{aligned} \mathfrak{J}(g) &= \sum_{m,n=1}^{\infty} \sum_{\ell=0}^{\infty} g^{2\ell+m+n-2} \mathfrak{J}_{m,n}^{(\ell)} = \mathfrak{J}_0 + g \mathfrak{J}_{1,2}^{(0)} + g \mathfrak{J}_{2,1}^{(0)} + g^2 \mathfrak{J}_{1,1}^{(1)} + g^2 \mathfrak{J}_{1,3}^{(0)} + g^2 \mathfrak{J}_{2,2}^{(0)} + g^2 \mathfrak{J}_{3,1}^{(0)} + \dots \\ &= \int \mathcal{A} \end{aligned} \quad (3)$$

At order g , the generators relate A_n to A_{n-1} . The first term in (3) is the free generator $\mathfrak{J}_0 = \mathfrak{J}_{1,1}^{(0)}$. The contributions $\mathfrak{J}_{1,2}^{(0)}, \mathfrak{J}_{1,3}^{(0)}$ increase the number of legs by one or two, respectively, while $\mathfrak{J}_{2,1}^{(0)}, \mathfrak{J}_{3,1}^{(0)}$ decrease it. In order to construct an invariant of the symmetry algebra, write all amplitudes in terms of a generating functional $\mathcal{A}[J]$ with sources $J(\Lambda_k)$:

$$\mathcal{A}(g)[J] = \sum_{n=4}^{\infty} \frac{g^{n-2}}{n} \int d\Lambda A_n(\Lambda) \text{Tr}(J(\Lambda_1) \dots J(\Lambda_k)) = \frac{g^2}{4} \int \mathcal{A}_4 + \frac{g^3}{5} \int \mathcal{A}_5 + \frac{g^4}{6} \int \mathcal{A}_6 + \dots$$

Here, $\Lambda_k = (\lambda_k, \bar{\lambda}_k, \eta_k)$ and $\Lambda = (\Lambda_1, \dots, \Lambda_k)$.

Collecting contributions with equal numbers of legs and powers in g , the invariance equation (2) takes the form:

$$\mathfrak{J}(g)\mathcal{A}(g) = 0 \Rightarrow 0 = \int \mathcal{A}_4 + \int \mathcal{A}_5 + \int \mathcal{A}_6 + \dots$$

Holomorphic Anomaly

Indeed, corrections to the free representation of superconformal symmetry are necessary: In Lorentz signature, tree level scattering amplitudes are not exactly invariant under the free representation. This is due to the holomorphic anomaly [2] contributing when two external momenta become collinear,

$$\frac{\partial}{\partial \bar{\lambda}^{\dot{a}}} \frac{1}{\langle \lambda, \mu \rangle} = \pi \delta^2(\langle \lambda, \mu \rangle) \varepsilon_{\dot{a}b} \bar{\mu}^b.$$

For instance, the action of the free generator $\bar{\mathcal{S}}_0$ on MHV amplitudes reduces to an expression which does not vanish in the collinear limit of two external momenta

$$(\bar{\mathcal{S}}_0)_{\dot{a}}^B A_n^{\text{MHV}} = -\pi \sum_{k=1}^n \varepsilon_{\dot{a}b} (\bar{\lambda}_{k-1}^b \eta_k^B - \bar{\lambda}_k^b \eta_{k-1}^B) \frac{\delta^2(\langle \lambda_{k-1}, \lambda_k \rangle) \delta^4(P) \delta^8(Q)}{\langle 12 \rangle \dots \langle k-1, k \rangle^0 \dots \langle n1 \rangle}. \quad (4)$$

Tree Level Correction

At tree level, only the supersymmetry generators \mathcal{S} , $\bar{\mathcal{S}}$ and the generator of special conformal transformations \mathfrak{K} receive corrections (the subscript refers to the helicity):

$$\mathcal{S} = \mathcal{S}_0 + g \mathcal{S}_+, \quad \bar{\mathcal{S}} = \bar{\mathcal{S}}_0 + g \bar{\mathcal{S}}_+, \quad \mathfrak{K} = \mathfrak{K}_0 + g \mathfrak{K}_- + g \mathfrak{K}_+ + g^2 \mathfrak{K}_{++},$$

where e.g. $\bar{\mathcal{S}}_+$ cancels the contribution (4) of the free generator $\bar{\mathcal{S}}_0$ and is given by [4]

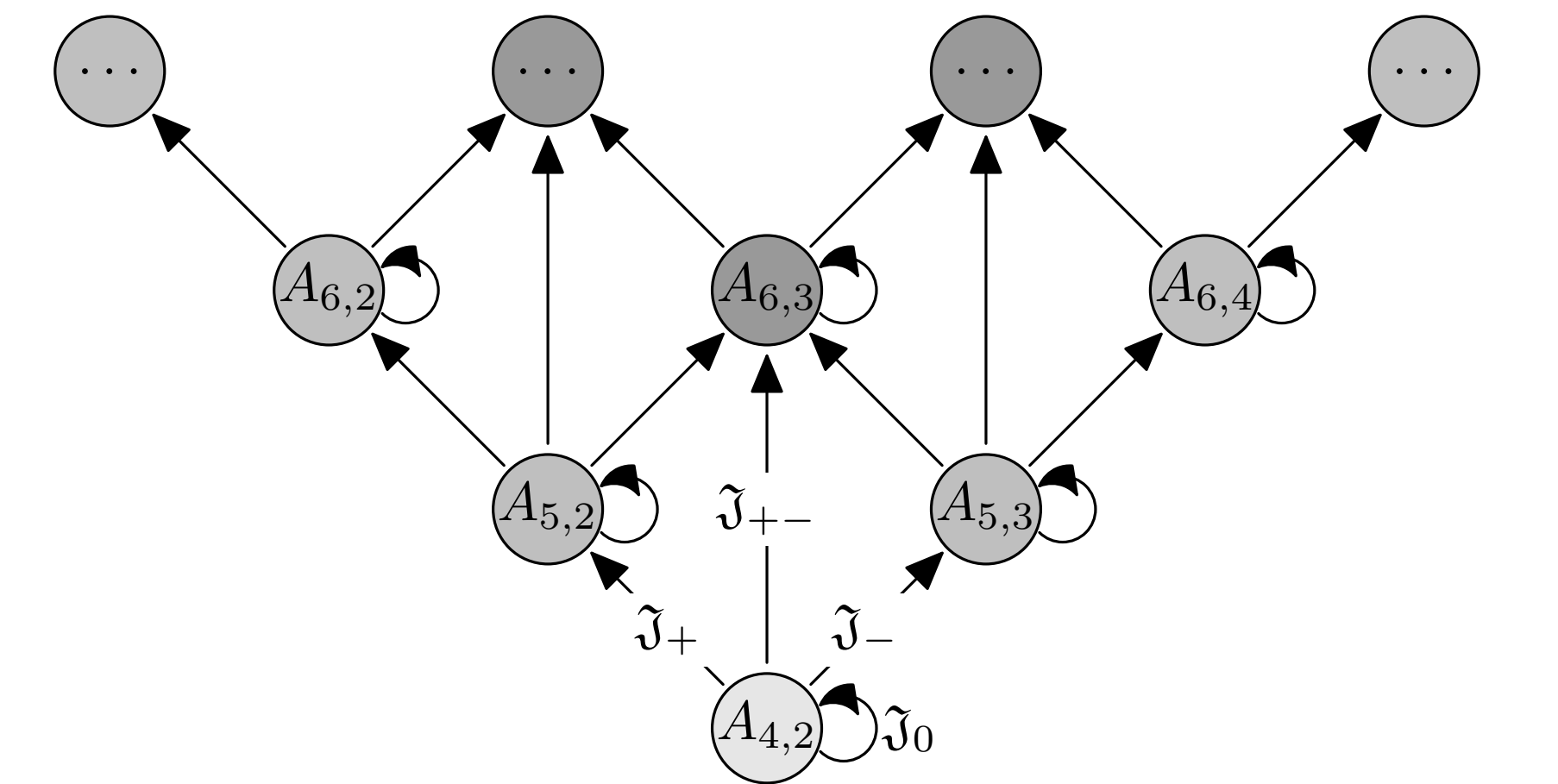
$$(\bar{\mathcal{S}}_+)_{\dot{a}}^B J(\Lambda) = \pi \int d^4 \bar{\eta}' da d\varphi e^{3i\varphi} \varepsilon_{\dot{a}\dot{\gamma}} \bar{\lambda}_1^{\dot{\gamma}} \bar{\eta}_2^B [J(\Lambda_1), J(\Lambda_2)], \quad \begin{aligned} \lambda_1 &= \lambda e^{i\varphi} \sin \alpha, & \bar{\eta}_1 &= (\bar{\eta} \sin \alpha + \bar{\eta}' \cos \alpha) e^{-i\varphi}, \\ \lambda_2 &= \lambda \cos \alpha, & \bar{\eta}_2 &= \bar{\eta} \cos \alpha - \bar{\eta}' \sin \alpha. \end{aligned}$$

With similar corrections to the generators $\mathcal{S}_0, \mathfrak{K}_0$, the superconformal algebra closes up to gauge transformations,

$$[\mathfrak{J}_a, \mathfrak{J}'_b] \sim \mathcal{G}_{ab}, \quad \mathcal{G}_{ab} J(\Lambda) \sim [\partial_a \partial_b J(0), J(\Lambda)].$$

All Amplitudes

Employing the BCFW recursion relations [5], generic amplitudes show a specific scaling behaviour when two external momenta become collinear. Acting with the generators $\mathcal{S}_0, \bar{\mathcal{S}}_0, \mathfrak{K}_0$ shows the corrections from the MHV case to render the set of all amplitudes invariant. At tree level, the invariance equation generalises to [4]:

$$\mathfrak{J}_0 A_{n,k} + \mathfrak{J}_+ A_{n-1,k} + \mathfrak{J}_- A_{n-1,k-1} + \mathfrak{J}_{+-} A_{n-2,k-1} = 0.$$


Here, $A_{n,k}$ is the n -point amplitude with k negative-helicity particles; $A_n^{\text{MHV}} = A_{n,2}$ and $A_n^{\overline{\text{MHV}}} = A_{n,n-2}$. Dual superconformal symmetry is free of holomorphic anomalies and thus needs not be corrected. Hence it can be argued that the general tree-level amplitude is invariant under the Yangian symmetry of [6]; we are confident that the Yangian uniquely determines the amplitude.

Open Problems

- Show explicitly that tree amplitudes are uniquely determined by Yangian symmetry.
- Promote corrections to loop level. Does the exact algebra determine loop-level amplitudes?
- What about conformal inversions?
- Apply to $\mathcal{N} < 4$ gauge theories with matter?
- Similar effects for $E_{7(7)}$ in supergravity?

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