Problems with Poisson Approach

tracking example: European XFEL
collective uniform motion (CUM) approach
time dependent shape \(\rightarrow\) problems
dirty trick
some conclusions
P1 and P2 approach
simple example
point particle / gaussian bunch / discrete quadrupole
more conclusions
European XFEL

3rd harmonic RF

Gun

3rd harmonic RF

dogleg

4 accelerator modules

laser heater

4 accelerator modules

main linac

collimator

bunch compressors

SASE1

Q=1nC before BC0, SE=on

Q=1nC before BC2, SE=on

Q=1nC after BC0, SE=on

Q=1nC after BC2, SE=on
before BC0, Z=73m, 130 MeV, ~60A

longitudinal phase space

≈ 8 MeV

≈ 6.5 keV

laser heater on !!!
after BC0, Z=80m, 130 MeV, ~150A

longitudinal phase space

slice energy spread

≈ 8 MeV

≈ 16 keV
before BC1, Z=159m, 700 MeV, ~150A

longitudinal phase space

∀ 28 MeV

slice energy spread

∀ 16 keV
after BC1, Z=181m, 700 MeV, ~650A

longitudinal phase space

≈ 28 MeV

slice energy spread

≈ 70 keV
before BC2, Z=370m, 2.4 GeV, ~650A

longitudinal phase space

≈ 38 MeV

slice energy spread

≈ 70 keV
after BC2, Z=393m, 2.4 GeV, ~5kA

≈ 38 MeV
after L3, Z=1628m, 14 GeV, \( \sim \)5kA

\[ \approx 35 \text{ MeV} \]
after L3, Z=1628m, 14 GeV, ~5kA

bunch shape after L3

![Graph showing bunch shape](image-url)
Collective Uniform Motion (CUM) Approach

Poisson solver $\rightarrow$  $E$

$B = \frac{1}{c^2} \mathbf{v}_c \times \mathbf{E}$

Lorentz force $\mathbf{F}_v = q \left( 1 + \frac{1}{c^2} \mathbf{v}_v \times \mathbf{v}_c \times \right) \mathbf{E}_v$

in particular $\mathbf{v}_v \parallel \mathbf{v}_c \rightarrow$ strong suppression of transverse force

$\mathbf{v}_v = \mathbf{v}_c \rightarrow \mathbf{F}_\perp = \frac{1}{\gamma^2} q \mathbf{E}_\perp$

$F_\parallel = qE_\parallel \sim \frac{1}{\gamma^2}$

usually $|E_\parallel| \ll \|E_\perp\|$
Time Dependent Shape + Long Bunch Approximation

long bunch estimation:

$$E_z \quad 2.4 \rightarrow 14 \text{ GeV}: \sim 10 \text{ MV} / 1000\text{m}$$

$$E_x \approx \frac{Z_0 I \cdot x}{2\pi \sigma_r \cdot \sigma_r} \quad \text{about} \quad 10 \text{ GV} / \text{m} \quad \text{for} \quad I = 5 \text{kA}, \sigma_r = 30 \mu\text{m}$$

$$F_{v,\parallel} \approx qE_{v,z} + x'v \cdot \frac{qZ_0 I \cdot x_v}{2\pi \sigma_r \cdot \sigma_r}$$ 

strong 2\textsuperscript{nd} order effect

f.i. \quad x'v \sim 1\mu\text{rad} \rightarrow \sim 10 \text{kV} / \text{m}

effects of z & x components of same magnitude

z comp.: decreasing with energy, strong correlation in slice \rightarrow corr. energy spread

x comp.: \sim energy independent, weak correlation in slice \rightarrow uncorr. energy spread
slice correlated and uncorrelated angle \( x'_v = x'_{sc}(z_v) + \delta x'_v \)

\[
F_{v,\parallel} \approx qE_{v,z} + \frac{qZ_0 I}{2\pi\sigma^2} \cdot x'_{sc}(z_v) x_v + \frac{qZ_0 I}{2\pi\sigma^2} \cdot \delta x'_v x_v
\]

even the correlated part contributes to uncorrelated energy spread!

ewtreme case: if the bunch is infinitely long there is no slice-to-slice-interaction and one can calculate slice-self-interaction with better frames that are adjusted to the correlated angle

only the uncorrelated angel spread would contribute to the longitudinal field!

\[
F_{v,\parallel} \approx \frac{qZ_0 I}{2\pi\sigma^2} \cdot \delta x'_v x_v
\]
The Dirty Trick

Lorentz force, Poisson approach

\[ F_v = q \left( 1 + \frac{1}{c^2} v_v \times v_c \times E_v \right) \]

slice correlated and uncorrelated motion

\[ v_v = v_{sc}(z_v) + \delta v_v \]

modified force

\[ F_v = q \left( 1 + \frac{1}{c^2} (v_c + \delta v_v) \times v_c \times E_v \right) \]
First Conclusions

it is an empirical approach

problems with rollover part: very different motion of particles in the same slice

perhaps it is better to consider IUM (individual uniform motion, per particle)
needs other numerical method
try to avoid quadratic scaling of effort

but ...

even the IUM approach is empirical/questionable!
full Maxwell-approaches (LW or PDE) could be better

... high effort, ??? gain of accuracy

!!! the Poisson approach can do better
Two Approaches for Tracking

EoM with E&B:
\[
\frac{d}{dt} \mathbf{r}_v = \mathbf{v} (\mathbf{p}_v) \\
\frac{d}{dt} \mathbf{p}_v = q \left[ \mathbf{E} + \mathbf{v} \times \mathbf{B} \right]
\]

Poisson approach:
\[
\begin{bmatrix}
\partial_x^2 + \partial_y^2 + \gamma_0^2 \partial_z^2
\end{bmatrix} V = - \frac{\rho}{\varepsilon} \\
V \\
\mathbf{A} = e_z c^{-1} \beta_0 V \\
\mathbf{B} = \nabla \times \mathbf{A} \\
\mathbf{E} = -\nabla V - \partial_z \mathbf{A} = -\nabla V + \beta_0 \partial_z \mathbf{A}
\]

EoM with V&A in canonical coordinates:
\[
\frac{d}{dt} \mathbf{r}_v = \mathbf{v} (\mathbf{P}_v - \mathbf{A} (\mathbf{r}_v)) \\
\frac{d}{dt} \mathbf{p}_v = -q \nabla [V - \mathbf{v} \cdot \mathbf{A}]
\]
\[
\begin{bmatrix}
\partial_x^2 + \partial_y^2 + \gamma_0^2 \partial_z^2
\end{bmatrix} V = - \frac{\rho}{\varepsilon} \\
V \\
\mathbf{A} = e_z c^{-1} \beta_0 V
\]
Simple Example

infinite charged plate in uniform motion \( \rho(x, y, z, t) = \rho(x - v_x t) \), \( \rho(u) = \begin{cases} \rho_0 & \text{if } |u| < a \\ 0 & \text{otherwise} \end{cases} \)

exact solution

\( V = V(x - v_x t) \)

\( V(u) = \frac{\rho_0}{\sqrt{1 - \beta_x^2}} \begin{cases} -\frac{u^2}{a^2 - 2a|u|} & \text{if } |u| < a \\ \text{otherwise} \end{cases} \)

\( cA = [\beta_x e_x + \beta_z e_z] V(x - v_x t) \)

\( cB = -\beta_z e_y V'(x - v_x t) \)

\( E = [(\beta_x^2 - 1)e_x + \beta_x \beta_z e_z] V'(x - v_x t) \)
**Simple Example**

infinite charged plate in uniform motion\[ \rho(x, y, z, t) = \rho(x - v_x t), \quad \rho(u) = \begin{cases} \rho_0 & |u| < a \\ 0 & \text{otherwise} \end{cases} \]

exact solution

\[ V = V(x - v_x t) \]

\[ V(u) = \frac{\rho_0}{1 - \beta_x^2} \frac{-u^2}{a^2 - 2a|u|} \begin{cases} & \text{if } |u| < a \\ & \text{otherwiese} \end{cases} \]

\[ cA = [\beta_x e_x + \beta_z e_z] V(x - v_x t) \]

\[ cB = -\beta_z e_y V'(x - v_x t) \]

\[ E = [\beta_x^2 - 1] e_x + \beta_x \beta_z e_z V'(x - v_x t) \]

Poisson solution, for \( \beta_0 = \beta_z e_z \neq \beta \)

\[ V_p = V_p(x - v_x t) \]

\[ V_p(u) = \frac{\rho_0}{\epsilon} \frac{-u^2}{a^2 - 2a|u|} \begin{cases} & \text{if } |u| < a \\ & \text{otherwiese} \end{cases} \]

\[ cA_p = \beta_z e_z V_p(x - v_x t) \]

\[ cB_p = -\beta_z e_y V_p'(x - v_x t) \]

\[ E_p = -e_x V_p'(x - v_x t) \]
Simple Example

tracking

parameters:
\[ a = 1 \text{ mm} \]
\[ \rho_0 = 1 \text{ C/m}^3 \]
\[ \gamma_0 = 10 \]
\[ \beta_x = \beta_0 \sin(0.1/\gamma_0) \]
\[ \beta_z = \beta_0 \cos(0.1/\gamma_0) \]

initial condition:
\[ t_0 = 0 \]
\[ x = 0.5a \]
\[ z = 0 \]
\[ v_x = 0 \]
\[ v_z = \beta_0 c \]
What is different?
Why is VA-method better?

It is not because coordinates are canonical!
It is because the field approximation is better:

still Poisson, but

**Two Approaches for Field Calculation**

\[
\left[ \partial^2_x + \partial^2_y + \gamma_0^2 \partial^2_z \right] V = - \rho / \epsilon
\]

\[
\downarrow
\]

\[
V
\]

\[
A = e_z c^{-1} \beta_0 V
\]

\[
B = \nabla \times A
\]

\[
E = -\nabla V - \partial_t A = -\nabla V + \beta_0 \partial_z A
\]

= “P1 approach”

\[
\partial_t A = -v_z \partial_z A \text{ assumes } A = A(z-v_z t)
\]

use the same approach as VA-method: = “P2 approach”

\[
E = -\nabla V - \partial_t A = -\left[ \nabla + e_z c^{-1} \beta_0 \partial_t \right] V
\]
Again: Simple Example

tracking

parameters:

\[ a = 1 \text{ mm} \]
\[ \rho_0 = 1 \text{ C/m}^3 \]
\[ \gamma_0 = 10 \]

\[ \beta_x = \beta_0 \sin(0.1/\gamma_0) \]
\[ \beta_z = \beta_0 \cos(0.1/\gamma_0) \]

initial condition:

\[ t_0 = 0 \]
\[ x = 0.5a \]
\[ z = 0 \]
\[ v_x = 0 \]
\[ v_z = \beta_0 c \]
**Point Particle**

**exact (UM)**

\[
E = \frac{q}{4\pi \varepsilon} \left( \frac{\mathbf{r} \gamma_q}{r^2 + \left( \mathbf{r} \cdot \frac{\mathbf{p}_q}{m_0 c^2} \right)^2} \right)^{3/2}
\]

\[
cB = \frac{\mathbf{p}_q}{m_0 c^2} \times \mathbf{E}
\]

**P1 approach (\(\partial_t \mathbf{A} = -\nu_z \partial_z \mathbf{A}\))**

\[
E_1 = \frac{q}{4\pi \varepsilon} \left( \frac{\mathbf{r} \gamma_0}{x^2 + y^2 + \gamma_0^2 z^2} \right)^{3/2}
\]

\[
cB_1 = \beta_0 \mathbf{e}_z \times \mathbf{E}_1
\]

**P2 approach (\(\partial_t \mathbf{A}\))**

\[
E_2 = \frac{q \gamma_0}{4\pi \varepsilon} \frac{\mathbf{r} \cdot \mathbf{e}_z \beta_0 \left[ x \beta_x + y \beta_y + \gamma_0^2 z \left( \beta_z - \beta_0 \right) \right]}{x^2 + y^2 + \gamma_0^2 z^2}^{3/2}
\]

\[
cB_2 = \beta_0 \mathbf{e}_z \times \mathbf{E}_2 = cB_1
\]
Point Particle

normalized Lorentz force

\[ f = \frac{4\pi\varepsilon}{r^2} (E + v_r \times B) \]

\[ |v_t| = |v_s| = |v_0| \]

\[ v_t = v_s \]
Point Particle

normalized Lorentz force

\[ f = \frac{4\pi e}{r^2} (E + v_t \times B) \]

\[ v_t = v_s \]

\[ |v_t| = |v_s| = |v_0| \]

exact

P1

P2

\[ \psi = 0.5/\gamma \]
**Gaussian Bunch**

6D phase space distribution

\[ f(x, y, z, x', y', \delta) = f_x(x, x')f_y(y, y')f_z(z, \delta) \]

with

\[ f_x(x, x') = \frac{1}{2\pi \epsilon_x} \exp \left\{ \frac{x^2 \gamma_x + 2xx'\alpha_x + x'^2 \beta_x}{-2\epsilon_x} \right\} \]

\[ f_y(y, y') = \cdots \quad f_z(z, \delta) = \cdots \]

6D integration

\[ E = \frac{q}{4\pi \epsilon} \int \frac{r_q \gamma_q}{\left[ r_q^2 + \left[ r_q \cdot \frac{p_q}{m_q}\right]^2 \right]^{3/2}} f(q - r_q, p_q) dX_q \]

linearization for \( p_q = p_0 + \Delta p \) in denominator

\[ E \approx \frac{q}{4\pi \epsilon} \int r_q \gamma_q \left\{ \frac{1}{\left[ r_q^2 + \left[ r_q \cdot \frac{p_0}{m_q}\right]^2 \right]^{3/2}} - 3 \left[ r_q \cdot \frac{p_0}{m_q}\right] \left[ r_q \cdot \frac{\Delta p}{m_q}\right] \right\} f(q - r_q, p_q) dX_q \]

for \( \frac{1}{64 \gamma_0 \alpha^2} \gg \epsilon_{x,n}, \ldots \)

analytic integration of momenta coordinates \( \rightarrow 3D \) integral

\[ E \approx \frac{q}{4\pi \epsilon} \int r_q \gamma_q \left\{ \frac{1}{\left[ \ldots \right]^{3/2}} - \frac{2\gamma_q^2 z_q}{\left[ \ldots \right]^{5/2}} \left[ -\alpha_x \frac{x_q(x - x_q)}{\beta_y} - \alpha_y \ldots \right] \right\} f_r(q - r_q) dV_q \]
\[ \frac{\alpha_y}{\beta_y} = 3.1 \]

\[ \left| \frac{\alpha_x}{\beta_x} \right| \ll \left| \frac{\alpha_y}{\beta_y} \right| \]
\[ \frac{\alpha_y}{\beta_y} = 3.1 \]

\[
\left| \frac{\alpha_x}{\beta_x} \right| \ll \left| \frac{\alpha_y}{\beta_y} \right|
\]
gaussian bunches with different $\alpha/\beta$

\[ \frac{\alpha_y}{\beta_y} = +3.1 \]

\[ \frac{\alpha_y}{\beta_y} = -5.1 \]
outside field and inside fields are as from charge in uniform motion
only the $\mathbf{r}$ and $\mathbf{v}$ are different
Self-Field due to a Discrete Quadrupole
without radiation part

kick ~ offset

point to point as for individual uniform motion

\[ E = \frac{q}{4\pi \varepsilon} \sum_{\nu} \frac{\Delta r_{\nu} \gamma_{q}}{\left[\Delta r_{\nu}^2 + \left[\Delta r_{\nu} \cdot p_{\nu} \frac{1}{m_0 c}\right]^2\right]^{3/2}} \]

but: \( r_{\nu}, p_{\nu} \) are either the actual properties (after the quadrupole kick) or the properties without quadrupole; one has to distinguish if the retarded source is observed before or after the quadrupole; this depends on the location of the observer!
same (new) distribution, but different history

(distribution 3) drift --> discrete quad --> drift (distribution 4)

distribution 4
% HORIZONTAL
alphax = 0.2;
betax = 1.0;
% VERTICAL
alphay = 1.0;
betay = 0.322;

\[ \frac{\alpha_y}{\beta_y} = +3.1 \]
different bunches, but same history (before q.)

\[ \frac{\alpha_y}{\beta_y} = +3.1 \]

\[ \frac{\alpha_y}{\beta_y} = -5.1 \]
Some Conclusions 2

Xtrack uses P1 approach

results (uncorrelated longitudinal energy spread) are not satisfying

P1 approach does not consider transient shape variations

there are better methods f.i.

P2 approach
IUM (individual uniform motion)
Taylor expansion around $p_0$
are not perfect (f.i. long bunch in undulator, nun-IUM shape variations)

P2 is for free with $rP$-state-variables, needs $\partial V/\partial t$ with $rp$-state-variables

examples of IUM-type: infinite plate with transverse motion
point particle
6D gaussian bunch

--> P2 significantly better than P1, close to IUM
P2 slightly better than P1