Isogeometric Analysis and its Application to Cavity Simulation

J. Corno\textsuperscript{1,2,3}, C. de Falco\textsuperscript{3}, H. De Gersem\textsuperscript{2} and S. Schöps\textsuperscript{1,2}

\textsuperscript{1} Graduate School CE, TU Darmstadt, Darmstadt, Germany
\textsuperscript{2} Institut für Theorie Elektromagnetischer Felder, TU Darmstadt, Darmstadt, Germany
\textsuperscript{3} MOX-Modeling and Scientific Computing, Dipartimento di Matematica, Politecnico di Milano, Milano, Italy
Outline

1 Motivation
2 IGA framework
3 Cavity Simulation
4 Outlook and Conclusions
Motivation

TESLA cavities require high precision but at the same time they are highly sensitive to shape changes.

Precision in the geometry representation is paramount.

**GOAL:** Use IGA and CAD properties to create an efficient framework for the description of these (and more complicated) deformations.
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Isogeometric Analysis

**Aim:** "bridging the gap between Computer Aided Design (CAD) and Finite Element Method (FEM)".
Isogeometric Analysis

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Both refinement and curved elements cannot exactly represent even simple CAD geometries (e.g. conic sections).
Isogeometric Analysis

**Idea:** use the CAD basis function (B-Splines, NURBS) for the analysis too (isoparametric approach).

- Exact representation of geometries defined via CAD even on the coarsest level
- Higher accuracy (per DoF)
- Higher regularity of the solution
- Field solution can change the geometry exactly (mechanics)
- Same framework as Classical FEM (you can reuse your code!)

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CAD Curves

**Knot vector**: \( \Xi = [\xi_0, \ldots, \xi_{n+p+1}] \), \( \xi_i \in [0, 1] \).

**Basis functions of degree** \( p \): \( N_{i,p} \).

Basis functions are weighted by control points to create a curve:

\[
C(\xi) = \sum_{i=0}^{n} P_i N_{i,p}(\xi)
\]

\( C(\xi) \) can be seen as a map from the reference to the physical space

\[
C(\xi) : [0, 1] \rightarrow \mathbb{R}^3
\]
CAD Curves

Basis functions are responsible for the regularity
Control points are responsible for the shape
NURBS are obtained as rational splines
A 2D reference domain.
CAD Surfaces

\[ \xi_1 = [0, 0, 0, 1/2, 1, 1, 1] \]
\[ \xi_2 = [0, 0, 0, 1/3, 2/3, 1, 1, 1] \]

The reference domain is divided by the knot vectors.
A control net is built on the physical domain.
The control points act as weights for the basis functions and create the physical domain.
A NURBS curve in $\mathbb{R}^2$ is the projection of a B-Spline curve in $\mathbb{R}^3$.

A NURBS volume in $\mathbb{R}^3$ is the projection of a B-Spline volume in $\mathbb{R}^4$ although this is difficult to visualize.
IGA vs Classical FEM Mapping

Classical FEM uses one (polynomial) map for each element.
IGA vs Classical FEM Mapping

IGA uses one global (B-Spline or NURBS) map for the whole domain.
IGA vs Classical FEM Mapping

In Classical FEM the refinement is performed on the **physical** domain.
In IGA the refinement is performed on the reference domain.
Curve Deformation

\[ C^{(1)}(\xi) = \sum_{i=0}^{n} P_{i}^{(1)} N_{i,p}(\xi) \]
Curve Deformation

\[ C^{(1)}(\xi) = \sum_{i=0}^{n} P_i^{(1)} N_{i,p}(\xi) \]

\[ C^{(2)}(\xi) = \sum_{i=0}^{n} P_i^{(2)} N_{i,p}(\xi) \]
Curve Deformation

\[ C^{(1)}(\xi) = \sum_{i=0}^{n} P_{i}^{(1)} N_{i,p}(\xi) \]

\[ C^{(2)}(\xi) = \sum_{i=0}^{n} P_{i}^{(2)} N_{i,p}(\xi) \]

\[ C^{(3)}(\xi) = \sum_{i=0}^{n} \left( P_{i}^{(1)} + P_{i}^{(2)} \right) N_{i,p}(\xi) \]
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Pillbox Cavity - Lorentz Detuning

Pillbox cavity with radius $R$ and length $L$. To compute the detuned $TM_{010}$ frequency after the deformation due to Lorentz Detuning we solve:

\[ \nabla \times \left( \frac{1}{\mu_0} \nabla \times \mathbf{E} \right) = \omega^2 \epsilon_0 \mathbf{E} \]
\[ \nabla \cdot \left( 2\eta \nabla^{(S)} \mathbf{u} + \lambda \mathbf{I} \nabla \cdot \mathbf{u} \right) = 0 \]

with $\mathbf{H} = \nabla \times \mathbf{E}/(i\omega\mu_0)$ and $f = \omega/(2\pi)$.

The EM problem couples into the mechanical problem by the radiation pressure

\[ p = -\frac{1}{4} \epsilon_0 \mathbf{E}_n \mathbf{E}_n^* + \frac{1}{4} \mu_0 \mathbf{H}_t \mathbf{H}_t^*. \]
Convergence of the eigenfrequency for the deformed pillbox cavity ($R = 35$ mm, $L = 100$ mm, detuned frequency $f_0' = 3.278292919$ GHz).

GeoPDEs uses a naïve iterative procedure, while CST uses sensitivity analysis to predict the detuning.
Applying the same iterative procedure in CST leads to lower accuracy due to the exporting of the deformed geometry.
### Pillbox Cavity - Computational Effort

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Number of DoFs required to compute the accelerating mode in the pill-box cavity within a prescribed accuracy. Computational times listed refer to the solution of the eigenvalue problem with ARPACK.

The correct accelerating field should have $E_x = E_y = 0$ and only longitudinal component $E_z$.

The FEM method suffers from oscillations due to non axis-aligned elements, while the IGA solution is precisely determined.
Uncertain Pillbox Cavity

Pillbox cavity with uncertain radius $r$ close to the design value $r_d$. We solve Maxwell’s eigenproblem

$$\nabla \times (\mu_0^{-1} \nabla \times \mathbf{E}) = (2\pi f)^2 \varepsilon_0 \mathbf{E}$$

to compute the fundamental frequency $f_0$ and its sensitivity w.r.t changes in radius

$$f_0(r) = \frac{2.405c}{2\pi r}, \quad \frac{df_0(r)}{dr} = -\frac{2.405c}{2\pi r^2}$$

where $G \approx 2.405$ is the first zero of the Bessel function $J_0$.

We compare IGA (GeoPDEs) to FEM (CST MWS®)
Uncertain Pillbox Cavity

Local Sensitivity (FD)

Global Sensitivity (gPC, unif. dist.)
Uncertain TESLA Cavity

The sensitivity of the quantity of interest (frequency, $R/Q$, etc...) for the TESLA cavity has already been performed as functions of the geometric parameters [1-3].

**TESLA design parameters**


Uncertain TESLA Cavity

The sensitivity of the quantity of interest (frequency, $R/Q$, etc...) for the TESLA cavity has already been performed as functions of the geometric parameters [1-3].

We need to be able to study more "crazy" uncertainties!


1cell TESLA Cavity - Elliptic Deformation

Sensitivity of the first 16 eigenfrequency in the 1cell TESLA cavity due to an **elliptical deformation** of its cross section (gPC, 10x10 collocation grid).
TESLA Deformation - Cells Eccentricity
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Standard Deviation

\[ \sigma_9 \sim 2012815 \text{ Hz} \]
Conclusions

- Exact representation of CAD geometries through IGA
- Exploit CAD properties to describe changes of geometry
- Smaller matrices and good computational efficiency
- Higher smoothness of the computed solution
- Uncertainty Quantification on the cavity shape

Ongoing work

- Design and optimization of a new injection cavity for the S-DALINAC
- Inclusion of the couplers in the simulation through an IGA-FEM coupling
- Extension to other geometrical uncertainties
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Please ask questions now or contact

Jacopo Corno
corno@gsc.tu-darmstadt.de
http://www.graduate-school-ce.de/index.php?id=619