The standard model from the metric point of view

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Description of the standard model within the framework of noncommutative geometry. Discrete structure of spacetime even without quantum gravity.

$$\begin{array}{ccc} \text{commutative algebra} & \rightarrow & \text{noncommutative algebra} \\ \updownarrow & & \downarrow \\ \text{differential geometry} & & \text{noncommutative geometry} \end{array}$$

A geometry "without points", but the notion of distance is available via Connes formula. The metric information is encoded within the Dirac operator

$$ds = D^{-1}$$

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- Riemannian compact spin manifold M:

$$-i\gamma^\mu \partial_\mu \Longleftrightarrow \text{riemannian geodesic distance.}$$

- Fibre bundle P with connection:

$$-i\gamma^\mu(\partial_\mu + A_\mu) \Longleftrightarrow ?$$

Outline:

1. The spectral triple of the standard model
   spectral triple
   connection and the product of the continum by the discrete
   the standard model

2. Distance in noncommutative geometry

3. Fluctuations of the metric
   scalar fluctuation and the standard model
   gauge fluctuation and holonomy obtruction

4. Spectral distance on the circle

Conclusion: extra-dimensions from Pythagoras theorem

1. The spectral triple of the standard model

Spectral triple

$(\mathcal{A}, \mathcal{H}, D)$ with $\mathcal{A}$ a *-algebra (commutative or not), represented over an Hilbert space $\mathcal{H}$. $D = D^*$ is an operator on $\mathcal{H}$ satisfying a set of properties such that

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2. Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is a commutative *-algebra then (Connes reconstruction theorem)
   - $\mathcal{A} = C^\infty(M)$ where M is a compact oriented spin manifold with Dirac operator $D_s = D +$ torsion term,
   - there exists a unique riemannian structure on M such that the associated geodesic distance is $d(x, y) = \sup_{f \in \mathcal{A}} \{f(x) - f(y) \,/\, \|[D, f]\| \leq 1\}$.
   - the functional $S(D) \doteq ⨍ |D|^{-n+2}$ attains its mimimum for $D = D_s$ and is proportional to Euclidean Einstein-Hilbert action.

$$\begin{array}{ccc} \text{spectral triple with commutative } \mathcal{A} & \rightarrow & \text{spectral triple with noncommutative } \mathcal{A} \\ \updownarrow & & \downarrow \\ \text{Riemannian spin manifold} & & \text{noncommutative geometry} \end{array}$$

- Extends the notion of geometry beyond the scope of Riemannian geometry (but always Euclidean signature).
- The standard model fits well in this framework. The action functional yields the lagrangian of the standard model minimally coupled to Einstein-Hilbert gravity.
- Gives a geometrical interpretation to the Higgs field.

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- H: scalar field on M with value in $\mathcal{A}_I$ $\longrightarrow$ Higgs.
- A_μ: 1-form field with value in $Lie(U(\mathcal{A}_I))$ $\longrightarrow$ gauge field.

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The *covariant Dirac operator* $D_A = D + A + JAJ^{-1}$ inherits a scalar field component.

The standard model *(Chamseddine, Connes, Marcolli. 2006)*

$$\mathcal{A}_I = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$$
$$\mathcal{H}_I = \mathbb{C}^{96}$$

D_I is a 96×96 matrix with the masses of the fermions and the CKM matrix.

- Spectral action: the heat kernel expansion of Tr $\left(f(\frac{D_A}{\Lambda})\right)$ yields Einstein-Hilbert action (with euclidean signature) together with a Weyl term and the full lagrangian of the standard model.
- f appears only through $f_0 = f(0), f_k = \int_0^\infty f(v)v^{k-1}dv$ for $k = 2, 4$. Three new parameters physically related to the coupling contants at the unification scale, the gravitational constant and the cosmological constant.
- three predictions:

$$g_2 = g_3 = \sqrt{\tfrac{5}{3}} g_1$$

$$\sum_{\text{generations}} m_e^2 + m_\nu^2 + 3m_d^2 + 3m_u^2 = 8M_W^2$$

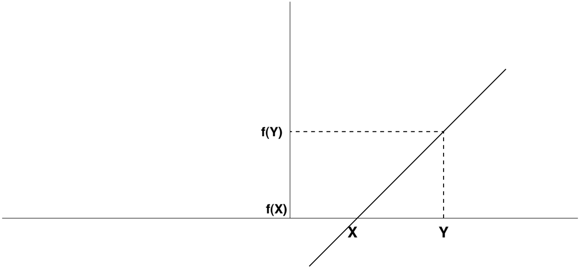
$$m_H \simeq 170\text{Gev}.$$

2. Distance in noncommutative geometry

Riemannian manifold M: $[\not\partial, f] = \overrightarrow{grad}\, f$

$$\sup_{f\in C^\infty(M)} \left\{ |f(x) - f(y)| \,/\, \left\| \overrightarrow{grad}\, f \right\| \leq 1 \right\} = d_{geo}(x, y).$$

Real line: $\sup_{f\in C^\infty(\mathbb{R})} \{ |f(x) - f(y)| \,/\, \|f'\| \leq 1 \} = |x - y|.$



- The upper bound is attained because there exists $f = f^*$ with $\left\| \overrightarrow{grad}\, f \right\| = 1$ everywhere on the geodesic (x, y), i.e $f(z) = d_{geo}(x, z)$.

Points are dual of functions. Gelfand duality,

$$\mathcal{P}(C^\infty(M)) \simeq M$$
$$\omega_x(f) = f(x)$$

with $\mathcal{P}(\mathcal{A})$ the pure states of $\mathcal{A}$ (normalized positive linear maps $C^\infty(M) \to \mathbb{C}$).

$$\boxed{d(\omega_x, \omega_y) \doteq \sup_{f \in C^\infty(M)} \left\{ |\omega_x(f) - \omega_y(f)| \,/\, \|[\partial\!\!\!/, f]\| \leq 1 \right\}}$$

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Definition of the distance that still makes sense for noncommutative $\mathcal{A}$.

$$\boxed{d(\omega_1, \omega_2) \doteq \sup_{a \in \mathcal{A}} \left\{ |\omega_1(a) - \omega_2(a)| \, / \, \|[D, a]\| \leq 1 \right\}}$$

- as soon as $[D, a]$ is bounded for all a, d is a distance between (pure) states.
- coherent with the classical case when $\mathcal{A} = C^\infty(M)$: $d = d_{geo}$,
- does not involve notions ill-defined in a quantum context (e.g. trajectories between points) but only spectral properties: *spectral distance*.

3. Fluctuations of the metric

The replacement $D \to D_A$ yields a *fluctuation of the metric* since

$$[D_A, a] = [D + H - i\gamma^\mu A_\mu, a] \neq [D, a].$$

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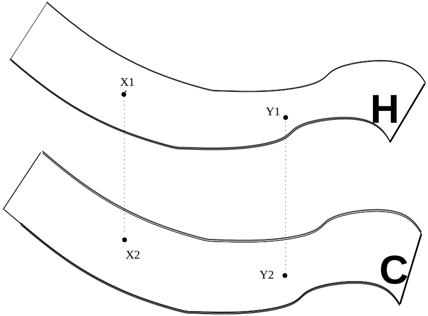
Scalar fluctuation: $A_\mu = 0, H \neq 0$ *(Wulkenhaar, P.M. 2001)*
$\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_I$ with $\mathcal{A}_I = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \Longrightarrow \mathcal{P}(\mathcal{A})$ is a two-sheet model



The spectral distance d coincides with the geodesic distance in $M \times [0,1]$ given by

$$\begin{pmatrix} g^{\mu\nu} & 0 \\ 0 & \left(|1+h_1|^2 + |h_2|^2\right) m_{\text{top}}^2 \end{pmatrix} \text{ where } \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \text{ is the Higgs doublet.}$$

Gauge fluctuation: $A_\mu \neq 0, H = 0$ *(P.M. 2005-07)*

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Example suggested by Connes (96)

$$\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_I \text{ with } \mathcal{A}_I = M_n(\mathbb{C}),\ D_I = 0.$$

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$\mathcal{P}(\mathcal{A})$ is a trivial bundle $P \xrightarrow{\pi} M$ with fiber $\mathbb{C}P^{n-1}$,

$$P \ni p = (x, \xi) = \xi_x, \quad \xi_x(a) = \langle \xi, a(x)\xi \rangle = \mathrm{Tr}(s_\xi a(x)).$$

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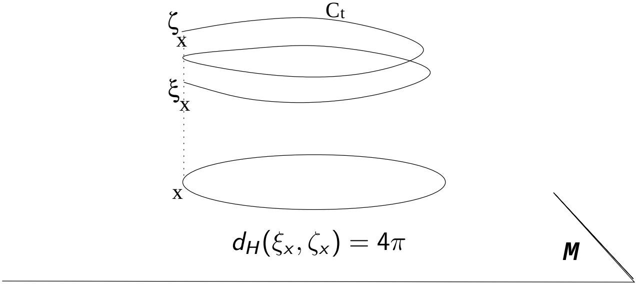
The part of D_A that does not commute with the representation is the covariant Dirac operator $-i\gamma^\mu(\partial_\mu + A_\mu)$ associated to to the connection.

The connection defines both a spectral distance d and an horizontal distance d_H:

$$T_pP = V_pP \oplus H_pP \Longrightarrow d_H(p,q) = \operatorname*{Inf}_{\dot{c}_t \in H_{c_t}P} \int_0^1 \|\dot{c}_t\| \, dt.$$



$$d_H(\xi_x, \zeta_x) = 4\pi$$

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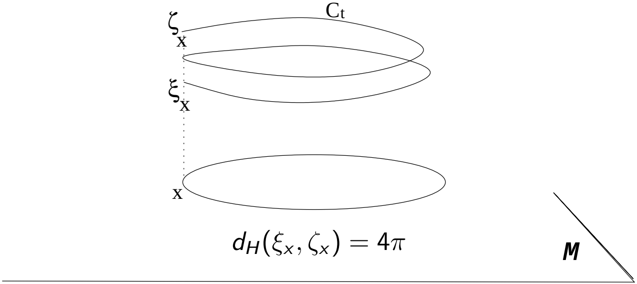


$$d \leq d_H$$

points at finite horizontal distance $\swarrow$ $\mathrm{Acc}(\xi_x)$

points at finite spectral distance $\searrow$ $\mathrm{Con}(\xi_x)$

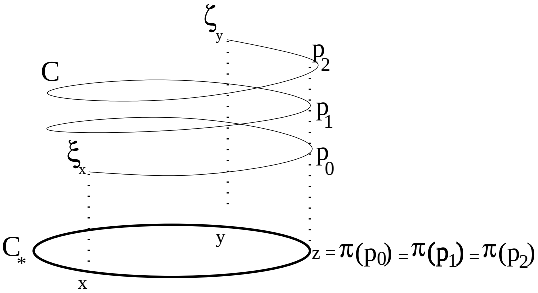
$$\boxed{\mathrm{Acc}(\xi_x) \subset \mathrm{Con}(\xi_x)}$$

Holonomy obstruction:

d_H plays for the bundle the same role as the d_{geo} for the manifold.

$$f(z) = \omega_z(f) = d_{\text{geo}}(x, z) \ \text{ reads } \ C_t(a) = d_H(\xi_x, C_t)$$

for any C_t in the minimal horizontal curve C between $\xi_x = C_0$, $\zeta_y = C_1$.



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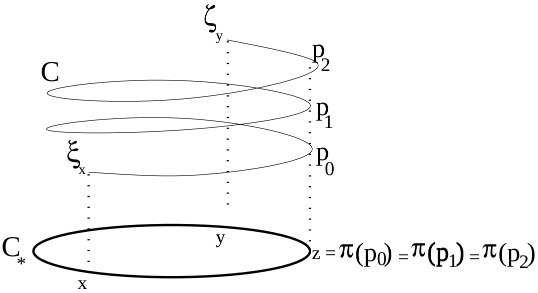

$$p_i(a) = \text{Tr}(s_{p_i} a(z)) = d_H(\xi_x, p_i).$$

- If more than n^2 points p_i, too many conditions on the single matrix $a(z)$! The spectral and horizontal distances cannot be equal.
- Is there a minimal horizontal curve with less than n^2 points p_i ?

4. Spectral distance on the circle

$\mathcal{A} = C^\infty(S^1, M_n(\mathbb{C})) \Longrightarrow$ pure states form a $\mathbb{C}P^{n-1}$ trivial bundle on S^1,

$$A = i \begin{pmatrix} \theta_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \theta_n \end{pmatrix}, \; \xi_x = \begin{pmatrix} V_1 \\ \vdots \\ V_n \end{pmatrix} \in \mathbb{C}P^{n-1}, \omega_j \doteq \int_0^{2\pi} \frac{\theta_1(t) - \theta_j(t)\; dt}{2\pi}.$$

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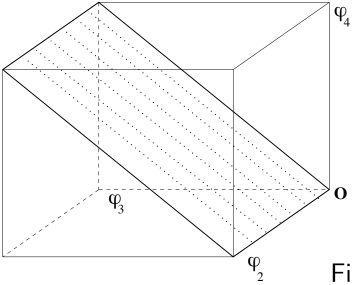
Topology of the fiber: $n = 4$, $\omega_3 = \omega_4$ irrational, $\omega_2 \in \mathbb{Q}$.

$\left\{ e^{i\varphi_i} |V_i|, i = 2, 3, 4 \right\}$ is a 3-torus inside $\mathbb{C}P^3$. Arg V_i fix a point.



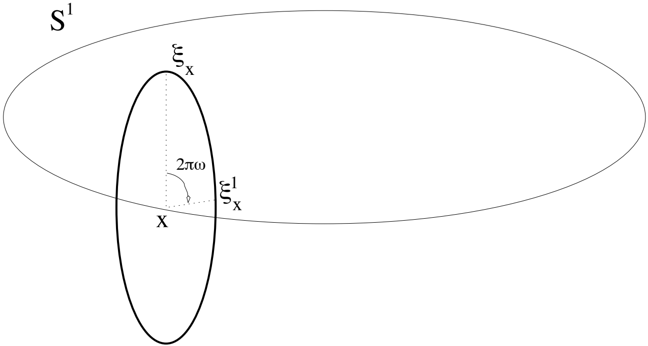
Fiberwise Con(ξ_x) is a 2-torus.

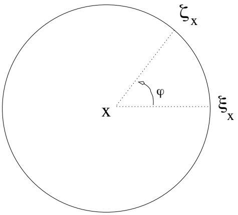
Acc(ξ_x) is at best dense in it. Globally, Con(ξ_x) is a 3-torus.

The shape of the fiber: $n = 2$

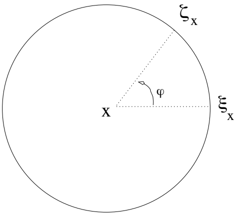
$n = 2 \Longrightarrow e^{i\varphi_2}|V_2|$ is a 1-torus inside the $\mathbb{C}P^1$ fiber.

- Fiberwise $\mathrm{Con}(\xi_x)$ is a 1-torus. Globally $\mathrm{Con}(\xi_x)$ is a 2-torus.
- Fiberwise $\mathrm{Acc}(\xi_x) = \mathrm{Hol}_x(\xi_x) = \{\xi_x^k, k \in \mathbb{Z}\}$ hence $d_H(\xi_x, \xi_x^k) = 2k\pi$.





$$\begin{cases} d_H(0,\varphi) = 2k\pi & \text{if} \quad \varphi = 2k\pi\omega \mod [2\pi] \\ d(0,\varphi) = C \sin\frac{\varphi}{2} & \text{with} \quad C = \frac{4\pi |V_1||V_2|}{|\sin \omega\pi|} \end{cases}$$



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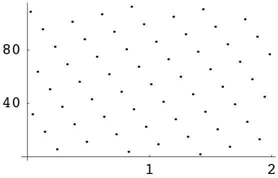

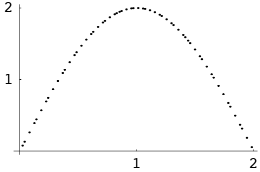

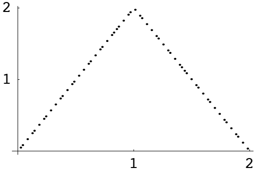

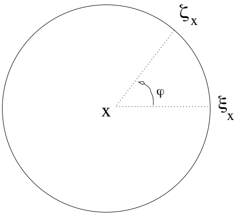

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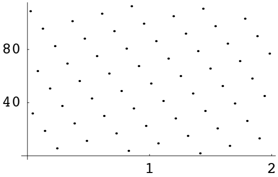
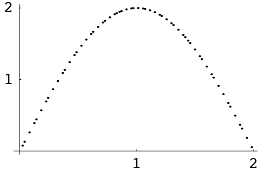
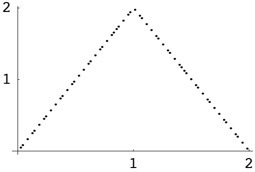
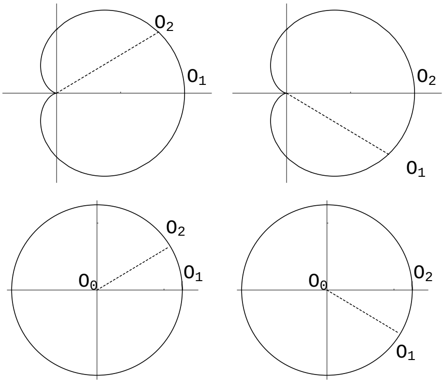






- No cutlocus for the distance function d: the fiber is smoother than a circle.

First interpretation: $d(0, \varphi)$ is the euclidean distance on the cardioid. But the latest is not invariant by rotation whereas d is.



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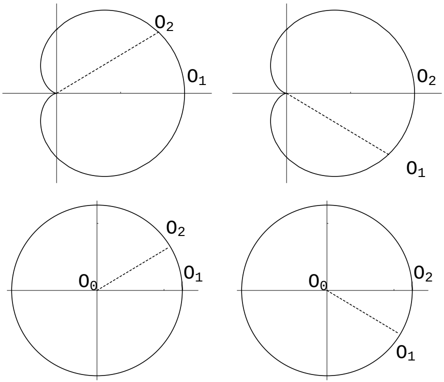
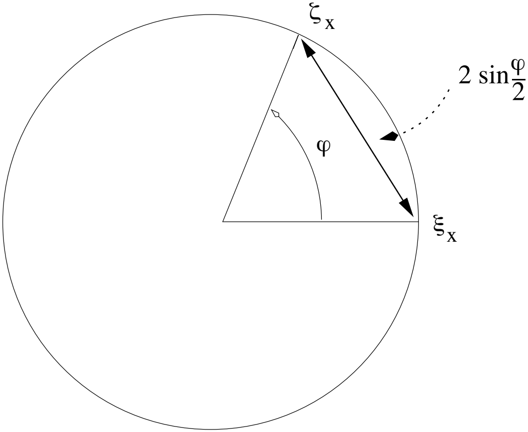


- With the spectral distance, everyone can equally pretend to be the center of the world.

Second interpretation: length of the segment in the disk



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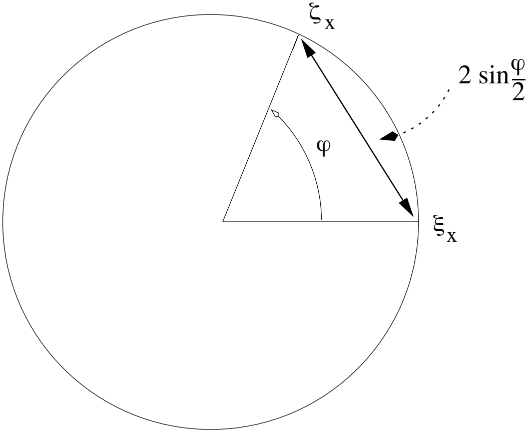

- The spectral distance “sees” the disk through the circle, in the same way as it sees between the sheets of the standard model.

Distance on the fiber for $n \geq 2$

$$\zeta_x \in \mathrm{Con}(\xi_x) = \begin{pmatrix} V_i & \forall i \in \mathrm{Far}_1 \\ e^{i\varphi_2} V_i & \forall i \in \mathrm{Far}_2 \\ \dots & \\ e^{i\varphi_{n_c}} V_i & \forall i \in \mathrm{Far}_{n_c} \end{pmatrix}, \varphi_j \in \mathbb{R}, \, j \in [2, n_c]$$

where Far_j are the classes of equivalence of $i \sim j$ iff $\omega_j = \omega_i \mod[2\pi]$.

$$d(\xi_x, \zeta_x) = \pi \mathrm{Tr}|S|$$

where S is the matrix with components

$$S_{ij} \doteq 2|V_i||V_j| \, \frac{\sin\left(\frac{\varphi_j - \varphi_i}{2}\right)}{\sin \pi(\omega_j - \omega_i)}.$$

- Not the Wilon loop but the trace of a matrix that contains the holonomy.

Conclusion: extra-dimensions from Pythagoras theorem

- $\underline{ds = D^{-1}}$:

$$D = \partial\!\!\!/ \otimes \mathbb{I}_I + \gamma^5 \otimes D_I \quad \Longrightarrow \quad D^2 = \partial\!\!\!/^2 \otimes \mathbb{I}_I + \mathbb{I}_E \otimes D_I^2$$

$$\Longrightarrow \quad ds^{-2} = ds_M^{-2} + ds_I^{-2} : \text{Pythagoras}^{-1}$$

However in the standard model

$$\begin{pmatrix} g^{\mu\nu} & 0 \\ 0 & \left(|1+h_1|^2 + |h_2|^2\right) m_{\text{top}}^2 \end{pmatrix} \Longrightarrow ds^2 = ds_M^2 + ds_I^2 : \ \text{Pythagoras}$$

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$$\begin{pmatrix} g^{\mu\nu} & 0 \\ 0 & \left(|1+h_1|^2 + |h_2|^2\right) m_{\text{top}}^2 \end{pmatrix} \Longrightarrow ds^2 = ds_M^2 + ds_I^2 : \ \text{Pythagoras}$$

Simple solution:

$$D_I^2 = |m|^2 \mathbb{I}_I \Longrightarrow D^2 = (\partial\!\!\!/^2 + |m|^2 \mathbb{I}_E) \otimes \mathbb{I}_I = (\gamma^a \partial_a)^2 \otimes \mathbb{I}_I$$

where $\gamma^a = \left\{\gamma^\mu, |m|\gamma^5\right\}$ and ∂_a is a “discrete derivative”.

Conclusion: extra-dimensions from Pythagoras theorem

- $\underline{ds = D^{-1}}$:

$$D = \partial\!\!\!/ \otimes \mathbb{I}_I + \gamma^5 \otimes D_I \quad \Longrightarrow \quad D^2 = \partial\!\!\!/^2 \otimes \mathbb{I}_I + \mathbb{I}_E \otimes D_I^2$$
$$\Longrightarrow \quad ds^{-2} = ds_M^{-2} + ds_I^{-2} : \text{Pythagoras}^{-1}$$

However in the standard model

$$\begin{pmatrix} g^{\mu\nu} & 0 \\ 0 & \left(|1+h_1|^2 + |h_2|^2\right) m_{\text{top}}^2 \end{pmatrix} \Longrightarrow ds^2 = ds_M^2 + ds_I^2 : \text{ Pythagoras}$$

Simple solution:

$$D_I^2 = |m|^2 \mathbb{I}_I \Longrightarrow D^2 = (\partial\!\!\!/^2 + |m|^2 \mathbb{I}_E) \otimes \mathbb{I}_I = (\gamma^a \partial_a)^2 \otimes \mathbb{I}_I$$

where $\gamma^a = \left\{\gamma^\mu, |m|\gamma^5\right\}$ and ∂_a is a "discrete derivative".

→ Illusion of extra-dimension come from Pythagoras theorem.

→ What is the equivalent for the disk ?

$$ds^2_{\text{disk}} = \text{function}(ds^2_{\text{circle}}, A) \;?$$

Outlook and references

- The spectral distance sees between the leaves of the horizontal foliation $\rightarrow$ should be relevant for the noncommutative torus.
- Topological effect due to $M = S^1$? Other basis for the bundle requires to know the number of selfintersecting points of the minimal horizontal curve. $\rightarrow$ work for (classical) subriemannian geometry.

- Discrete structure of space-time without talking of quantum gravity.
- Gauge fluctuation might make the distance on the $M_3(\mathbb{C})$ part of the standard model finite.

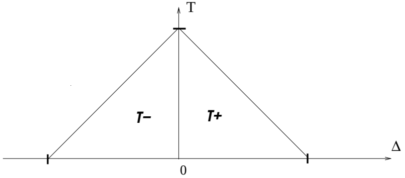
Spectral distance on S^1: math.OA/0703586, submitted to J. Func. Anal.

Carnot-Carathéodory vs NC-distance: Com.Math.Phys. **265** (2006) 585-616, or a non technical version, Cluj university press, *hep-th/0603051*.

Scalar fluctuation: with R. Wulkenhaar, J.Math.Phys. **43** (2002) 182-204.

Distance between the fibers for $n = 2$

$$d(\xi_x, \zeta_y) = \max_{\mathcal{T}_\pm} H_\xi(T, \Delta)$$



unit is $\min(\tau, 2\pi - \tau)$

where the sign is the one of $z_\xi \doteq |V_1|^2 - |V_2|^2$,

$$H_\xi(T, \Delta) \doteq T + z_\xi \Delta + W_1 \sqrt{(\tau - T)^2 - \Delta^2} + W_0 \sqrt{(2\pi - \tau - T)^2 - \Delta^2}$$

$$W_0 \doteq R \frac{|\sin(\frac{\varphi}{2})|}{|\sin \omega\pi|}, \ W_1 \doteq R \frac{|\sin(\omega\pi + \frac{\varphi}{2})|}{|\sin \omega\pi|}, \ R \doteq \sqrt{1 - z_\xi^2}.$$

- The element a that reaches the supremum has null diagonal at x, $\mathrm{Tr}(a(y)) = T$, $a_{11}(y) - a_{22}(y) = \Delta$.
- The maximum is reached for $T = 0$ or on the hypothenus.
- When $z_\xi = 0$ the maximum is reached at $\Delta = 0$.