The QCD static potential at NNNLL

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1. Definition

The potential is what to write in a Schrödinger equation

$$E\,\phi = \left(\frac{p^2}{m} + V(r)\right)\,\phi$$

In a full theory, V must come from a double expansion:

- a non-relativistic expansion $\sim p/m,\; rm$: $V \to V^{(0)} + V^{(1)}/m + \cdots$;
- an expansion in $E\,r$, since V is a function of r (or p at h.o. in the non-relativistic expansion): $V \to V+$ energy-dependent effects (e.g. Lamb-shift).

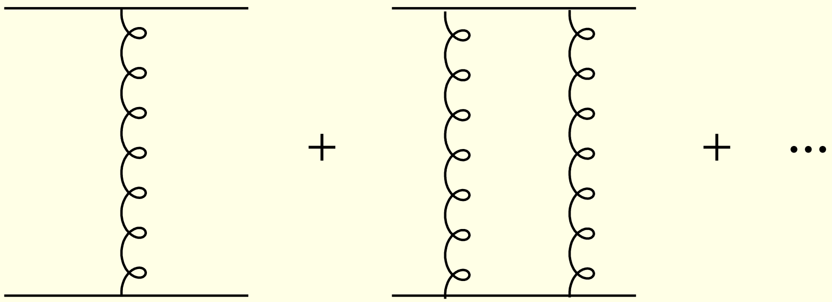
A potential V describes the interaction of a non-relativistic bound state, $p \sim mv$, $E \sim mv^2$, $v \ll 1$, once the expansions in mv/m and mv^2/mv have been exploited.

Non-relativistic scales in QCD

Near threshold:

$$E \approx 2m + \frac{p^2}{m} + \ldots \qquad \text{with} \quad v = \frac{p}{m} \ll 1$$

- The perturbative expansion breaks down when $\alpha_s \sim v$:



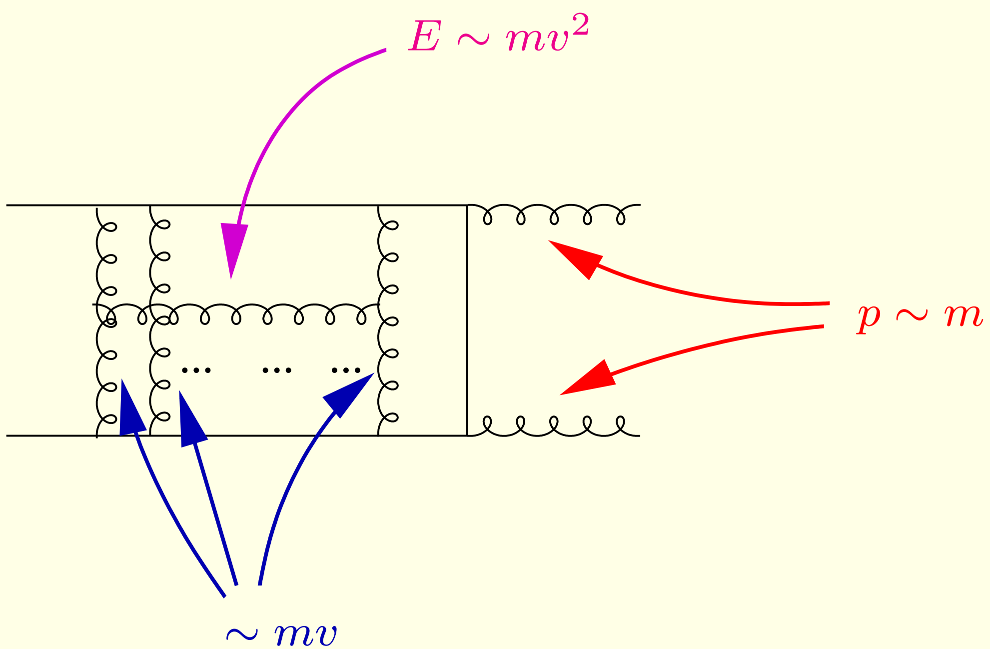
$$\approx \frac{1}{E - \left(\frac{p^2}{m} + V\right)}$$

$$\alpha_s \left(1 + \frac{\alpha_s}{v} + \ldots\right)$$

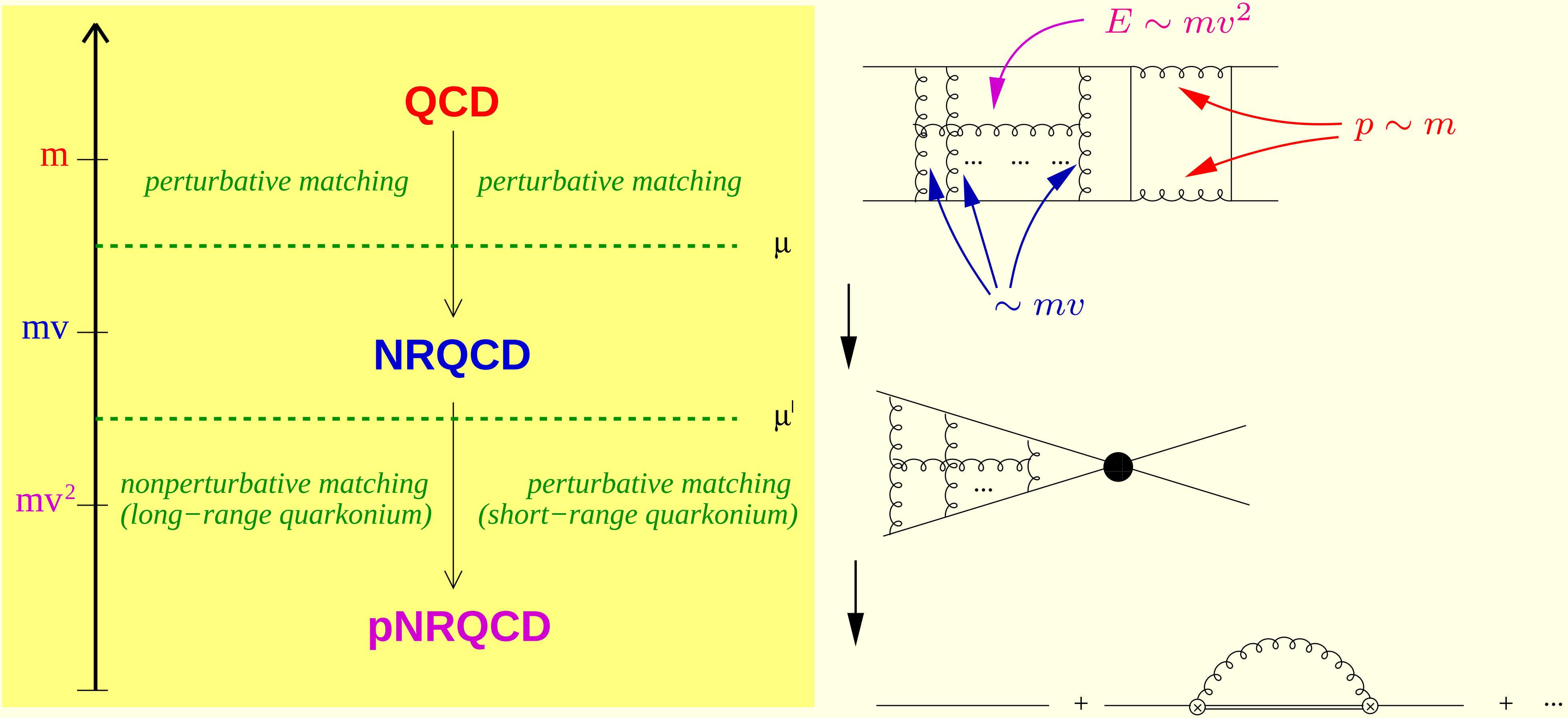
- The system is non-relativistic : $p \sim mv$ and $E = \dfrac{p^2}{m} + V \sim mv^2$.

Non-relativistic scales in QCD

Scales get entangled.



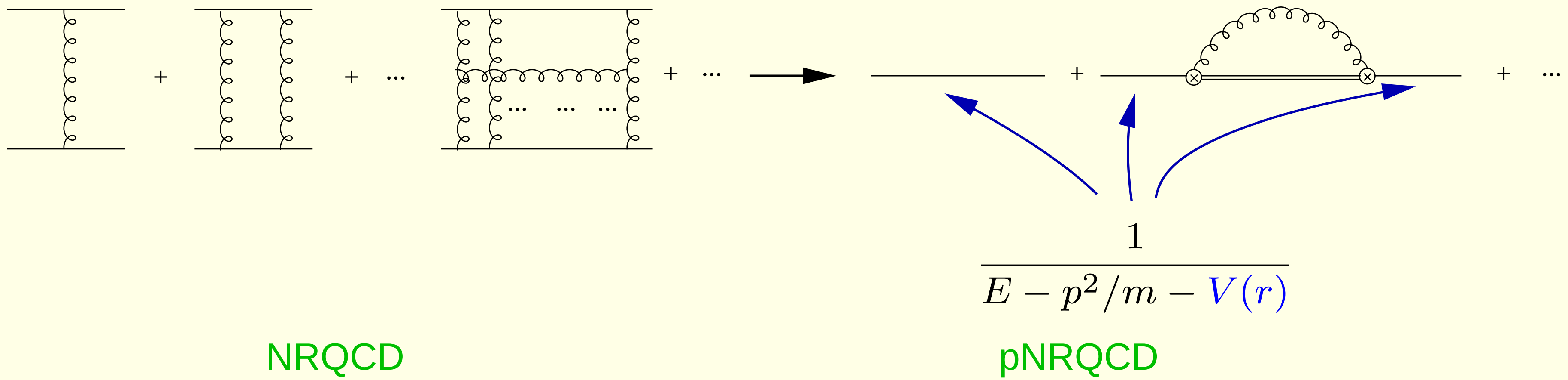
EFTs for systems made of two heavy quarks



- They exploit the expansion in v/ factorization of low and high energy contributions.
- They are renormalizable order by order in v.
- In perturbation theory (PT), RG techniques provide resummation of large logs.

pNRQCD

pNRQCD is obtained by integrating out modes associated with the scale $\frac{1}{r} \sim mv$



- The Lagrangian is organized as an expansion in $1/m$, r, and $\alpha_s(m)$:

$$\mathcal{L}_{\rm pNRQCD} = \sum_k \sum_n \frac{1}{m^k} \times c_k(\alpha_s(m/\mu)) \times V(r\mu', r\mu) \times O_n(\mu', \lambda)\, r^n$$

pNRQCD for $m\alpha_{\rm s} \gg \Lambda_{\rm QCD}$

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a} + \mathrm{Tr}\left\{ \mathrm{S}^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{m} - V_s \right) \mathrm{S} + \mathrm{O}^\dagger \left(iD_0 - \frac{\mathbf{p}^2}{m} - V_o \right) \mathrm{O} \right\}$$

LO in r

$$\theta(T)\, e^{-iTH_s}$$

$$\theta(T)\, e^{-iTH_o} \left(e^{-i\int dt\, A^{\rm adj}} \right)$$

pNRQCD for $m\alpha_{\rm s} \gg \Lambda_{\rm QCD}$

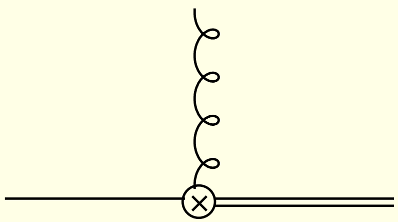
$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a} + {\rm Tr}\left\{ {\rm S}^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{m} - V_s \right) {\rm S} + {\rm O}^\dagger \left(iD_0 - \frac{\mathbf{p}^2}{m} - V_o \right) {\rm O} \right\}$$

LO in r

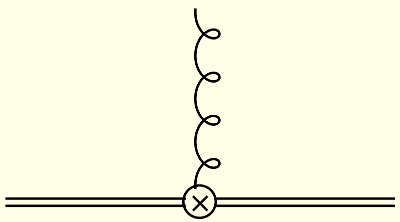
$$+ V_A {\rm Tr}\left\{ {\rm O}^\dagger \mathbf{r} \cdot g\mathbf{E}\, {\rm S} + {\rm S}^\dagger \mathbf{r} \cdot g\mathbf{E}\, {\rm O} \right\}$$
$$+ \frac{V_B}{2} {\rm Tr}\left\{ {\rm O}^\dagger \mathbf{r} \cdot g\mathbf{E}\, {\rm O} + {\rm O}^\dagger {\rm O} \mathbf{r} \cdot g\mathbf{E} \right\}$$
$$+ \cdots$$

NLO in r

pNRQCD for $m\alpha_{\rm s} \gg \Lambda_{\rm QCD}$



$O^\dagger \mathbf{r} \cdot g\mathbf{E}\, \mathrm{S}$



$O^\dagger \{\mathbf{r} \cdot g\mathbf{E},\, \mathrm{O}\}$

$$+V_A \,\mathrm{Tr}\left\{ \mathrm{O}^\dagger \mathbf{r} \cdot g\mathbf{E}\, \mathrm{S} + \mathrm{S}^\dagger \mathbf{r} \cdot g\mathbf{E}\, \mathrm{O} \right\}$$

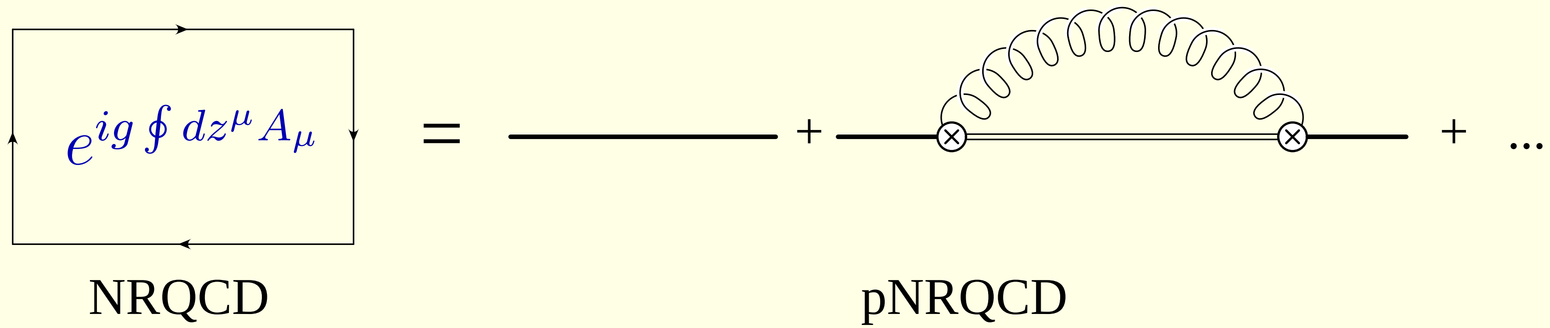
$$+\frac{V_B}{2} \,\mathrm{Tr}\left\{ \mathrm{O}^\dagger \mathbf{r} \cdot g\mathbf{E}\, \mathrm{O} + \mathrm{O}^\dagger \mathrm{O}\mathbf{r} \cdot g\mathbf{E} \right\}$$

NLO in r

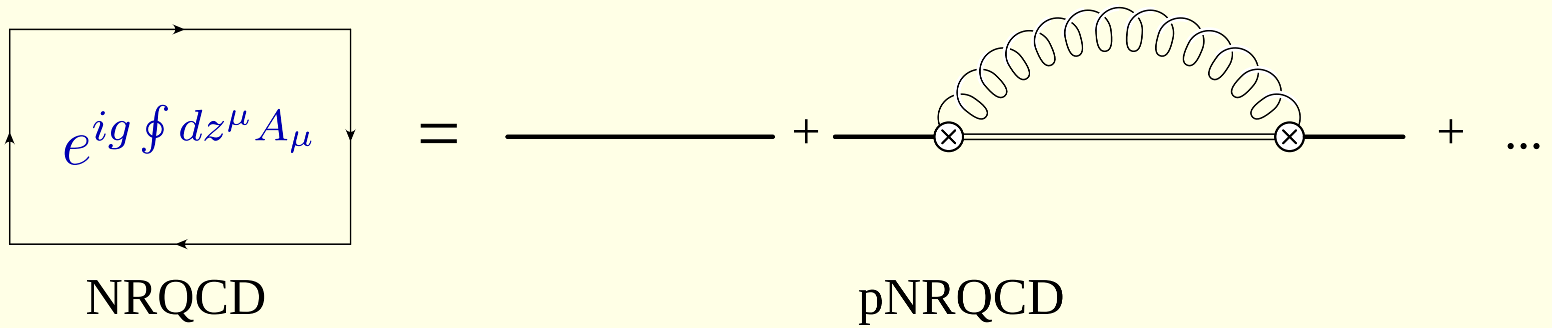
The potential is a Wilson coefficient of an EFT. In general, it undergoes renormalization, develops scale dependence and satisfies renormalization group equations, which allow to resum large logarithms.

2. Calculation

The Static Potential



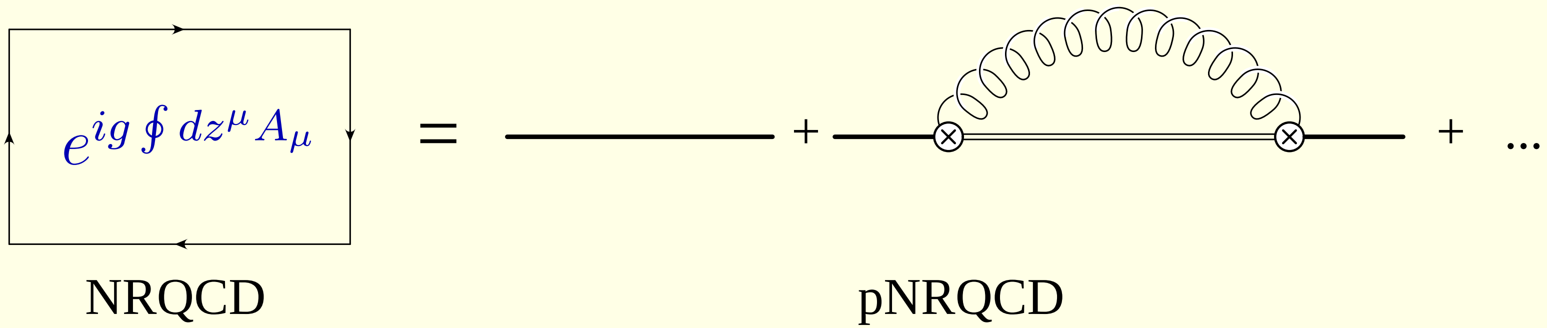
The Static Potential



$$\lim_{T\to\infty} \frac{i}{T} \ln \langle \Box \rangle = V_s(r,\mu) - i\frac{g^2}{N_c} V_A^2 \int_0^\infty dt\, e^{-it(V_o - V_s)} \langle \mathrm{Tr}(\mathbf{r}\cdot\mathbf{E}(t)\,\mathbf{r}\cdot\mathbf{E}(0))\rangle(\mu) + \ldots$$

ultrasoft contribution

The Static Potential



$$\lim_{T\to\infty} \frac{i}{T} \ln \left\langle \square \right\rangle = V_s(r,\mu) - i\frac{g^2}{N_c} V_A^2 \int_0^\infty dt\, e^{-it(V_o - V_s)} \langle \mathrm{Tr}(\mathbf{r}\cdot\mathbf{E}(t)\,\mathbf{r}\cdot\mathbf{E}(0))\rangle(\mu) + \ldots$$

ultrasoft contribution

* The μ dependence cancels between the two terms in the right-hand side:

$V_s \sim \ln r\mu, \ln^2 r\mu$, ...

ultrasoft contribution $\sim \ln(V_o - V_s)/\mu, \ln^2(V_o - V_s)/\mu, \ldots \ln r\mu, \ln^2 r\mu, \ldots$

Static Wilson loop

$$\lim_{T\to\infty} \frac{i}{T} \ln \left\langle \Box \right\rangle = -C_F \frac{\alpha_{\mathrm{s}}(1/r)}{r} \left[1 + a_1 \frac{\alpha_{\mathrm{s}}(1/r)}{4\pi} + a_2 \left(\frac{\alpha_{\mathrm{s}}(1/r)}{4\pi} \right)^2 + \ldots \right]$$

is known at two loops:

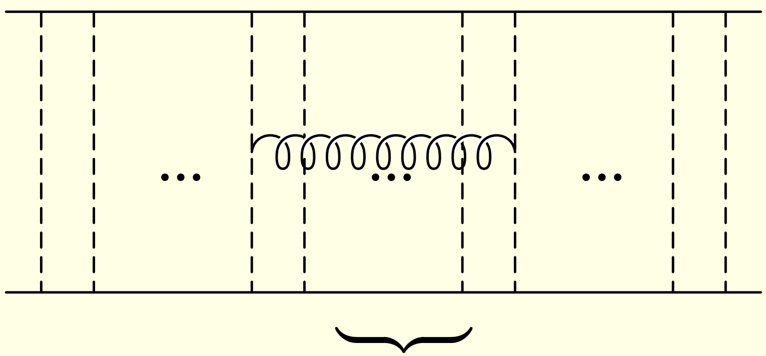
$$a_1 = \frac{31}{9} C_A - \frac{10}{9} n_f + 2\gamma_E \beta_0,$$

Billoire 80

$$\begin{aligned} a_2 \quad = \quad & \left(\frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3}\zeta(3) \right) C_A^2 - \left(\frac{899}{81} + \frac{28}{3}\zeta(3) \right) C_A n_f \\ & - \left(\frac{55}{6} - 8\zeta(3) \right) C_F n_f + \frac{100}{81} n_f^2 + 4\gamma_E \beta_0 a_1 + \left(\frac{\pi^2}{3} - 4\gamma_E^2 \right) \beta_0^2 + 2\gamma_E \beta_1 \end{aligned}$$

Schröder 99, Peter 97

Appelquist–Dine–Muzinich diagrams



$$= -\frac{C_F C_A^3}{12}\,\frac{\alpha_s}{r}\,\frac{\alpha_s^3}{\pi}\,\ln\left[\frac{C_A\alpha_s}{2r}\times r\right]$$

$$\sim \exp(-i(V_o - V_s)\,T)$$

Appelquist Dine Muzinich 78, Brambilla Pineda Soto Vairo 99

Static octet potential

$$\lim_{T\to\infty} \frac{i}{T} \ln \frac{\langle \square \rangle}{\langle \phi_{ab}^{\rm adj} \rangle} = \frac{1}{2N_c} \frac{\alpha_{\rm s}(1/r)}{r} \left[1 + b_1 \frac{\alpha_{\rm s}(1/r)}{4\pi} + b_2 \left(\frac{\alpha_{\rm s}(1/r)}{4\pi} \right)^2 + \dots \right]$$

Is known at two loops.

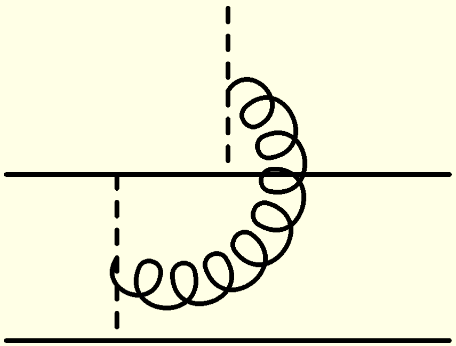
$$b_1 = a_1$$

$$b_2 = a_2 + C_A^2 (\pi^4 - 12\pi^2)$$

Kniehl et al 04

V_A

The first contributing diagrams are of the type:

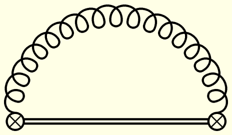


Therefore

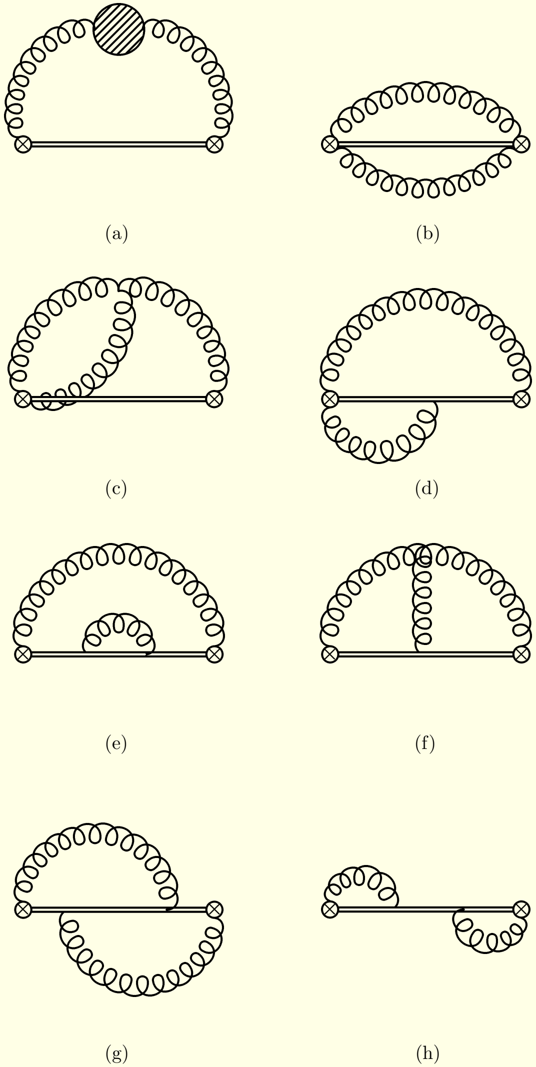
$$V_A(r, \mu) = 1 + \mathcal{O}(\alpha_s^2)$$

Chromoelectric field correlator: $\langle E(t)E(0)\rangle$

Is known at NLO.



LO



NLO

Eidemüller Jamin 97

Static singlet potential at $\mathrm{N^4LO}$

$$
\begin{aligned}
V_s(r,\mu) \;=\; & -C_F \frac{\alpha_s(1/r)}{r} \left[1 + a_1 \frac{\alpha_s(1/r)}{4\pi} + a_2 \left(\frac{\alpha_s(1/r)}{4\pi} \right)^2 \right. \\
& + \left(\frac{16\,\pi^2}{3} C_A^3 \, \ln r\mu + a_3 \right) \left(\frac{\alpha_s(1/r)}{4\pi} \right)^3 \\
& \left. + \left(a_4^{L2} \ln^2 r\mu + \left(a_4^L + \frac{16}{9} \pi^2 \, C_A^3 \beta_0 (-5 + 6 \ln 2) \right) \ln r\mu + a_4 \right) \left(\frac{\alpha_s(1/r)}{4\pi} \right)^4 \right]
\end{aligned}
$$

$$
\begin{aligned}
a_4^{L2} \;&=\; -\frac{16\pi^2}{3} C_A^3 \, \beta_0 \\
a_4^{L} \;&=\; 16\pi^2 C_A^3 \left[a_1 + 2\gamma_E \beta_0 + n_f \left(-\frac{20}{27} + \frac{4}{9} \ln 2 \right) \right. \\
& \qquad \left. + C_A \left(\frac{149}{27} - \frac{22}{9} \ln 2 + \frac{4}{9} \pi^2 \right) \right]
\end{aligned}
$$

Brambilla et al 99, 06

Static singlet potential at N^{4}LO

$$
\begin{aligned}
V_s(r,\mu) \;=\; & -C_F\,\frac{\alpha_{\rm s}(1/r)}{r}\left[1 + a_1\,\frac{\alpha_{\rm s}(1/r)}{4\pi} + a_2\left(\frac{\alpha_{\rm s}(1/r)}{4\pi}\right)^2\right. \\
& + \left(\frac{16\,\pi^2}{3}C_A^3\;\ln r\mu + a_3\right)\left(\frac{\alpha_{\rm s}(1/r)}{4\pi}\right)^3 \\
& \left. + \left(a_4^{L2}\ln^2 r\mu + \left(a_4^L + \frac{16}{9}\pi^2\,C_A^3\beta_0(-5+6\ln 2)\right)\ln r\mu + a_4\right)\left(\frac{\alpha_{\rm s}(1/r)}{4\pi}\right)^4\right]
\end{aligned}
$$

- The logarithmic contribution at N^{3}LO may be extracted from the one-loop calculation of the ultrasoft contribution;
- the single logarithmic contribution at N^{4}LO may be extracted from the two-loop calculation of the ultrasoft contribution.

Static energy at N^{4}LO

$$
\begin{aligned}
E_0(r) \quad = \quad & -\frac{C_F\alpha_{\rm s}(1/r)}{r}\Bigg\{1+\frac{\alpha_{\rm s}(1/r)}{4\pi}\left[a_1+2\gamma_E\beta_0\right] \\
& +\left(\frac{\alpha_{\rm s}(1/r)}{4\pi}\right)^2\left[a_2+\left(\frac{\pi^2}{3}+4\gamma_E^2\right)\beta_0^2+\gamma_E\left(4a_1\beta_0+2\beta_1\right)\right] \\
& +\left(\frac{\alpha_{\rm s}(1/r)}{4\pi}\right)^3\left[\frac{16\pi^2}{3}C_A^3\,\ln\frac{C_A\alpha_{\rm s}(1/r)}{2}+\tilde{a}_3\right] \\
& +\left(\frac{\alpha_{\rm s}(1/r)}{4\pi}\right)^4\left[a_4^{L2}\ln^2\frac{C_A\alpha_{\rm s}(1/r)}{2}+a_4^L\ln\frac{C_A\alpha_{\rm s}(1/r)}{2}+\tilde{a}_4\right] \\
& + \qquad \cdots \qquad \Bigg\}
\end{aligned}
$$

Renormalization group equations

$$\begin{cases} \mu \frac{d}{d\mu} \alpha_{V_s} = \frac{2}{3} \frac{\alpha_s}{\pi} V_A^2 \left[\frac{\alpha_{V_o}}{6} + \frac{4}{3} \alpha_{V_s} \right]^3 \left(1 + \frac{\alpha_s}{\pi}\, c\right) \\ \mu \frac{d}{d\mu} \alpha_{V_o} = \frac{2}{3} \frac{\alpha_s}{\pi} V_A^2 \left[\frac{\alpha_{V_o}}{6} + \frac{4}{3} \alpha_{V_s} \right]^3 \left(1 + \frac{\alpha_s}{\pi}\, c\right) \\ \mu \frac{d}{d\mu} \alpha_s = \alpha_s \beta(\alpha_s) \\ \mu \frac{d}{d\mu} V_A = 0 \\ \mu \frac{d}{d\mu} V_B = 0 \end{cases}$$

$$V_s = -C_F \frac{\alpha_{V_s}}{r}, \quad V_o = \frac{1}{2N_c} \frac{\alpha_{V_o}}{r}, \quad c = \frac{-5n_f + C_A(6\pi^2 + 47)}{108}.$$

Pineda Soto 00, Brambilla Garcia Soto Vairo 07

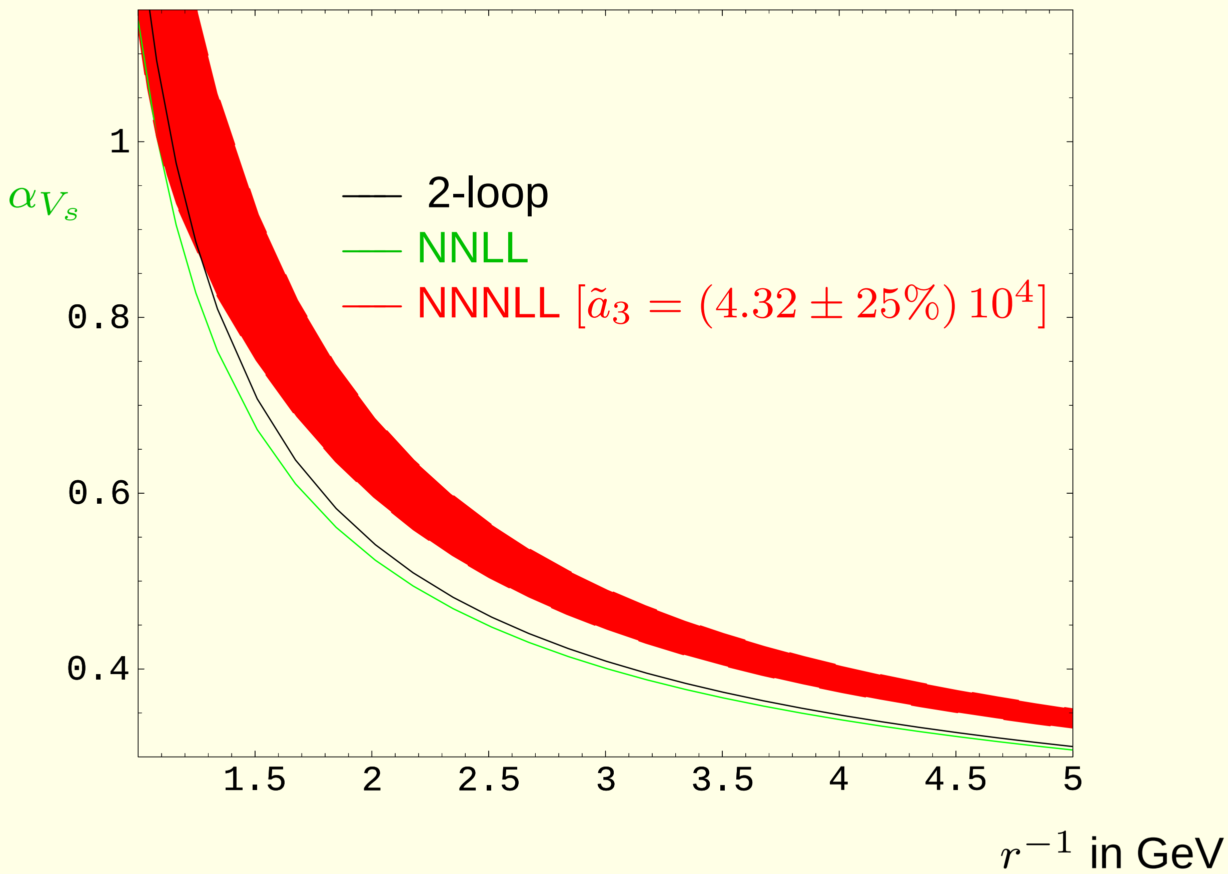
Static singlet potential at NNNLL

$$\alpha_{V_s}(\mu) = \alpha_{V_s}(1/r) + \frac{C_A^3}{6\beta_0}\alpha_s^3(1/r)\left\{\left(1+\frac{3}{4}\frac{\alpha_s(1/r)}{\pi}a_1\right)\ln\frac{\alpha_s(1/r)}{\alpha_s(\mu)}\right. \left.\left(\frac{\beta_1}{4\beta_0}-6c\right)\left[\frac{\alpha_s(\mu)}{\pi}-\frac{\alpha_s(1/r)}{\pi}\right]\right\}$$

$$V_s = -C_F\frac{\alpha_{V_s}}{r}, \quad a_1 = \frac{31}{9}C_A - \frac{10}{9}n_f + 2\gamma_E\beta_0, \quad c = \frac{-5n_f + C_A(6\pi^2+47)}{108}.$$

Brambilla Garcia Soto Vairo 07

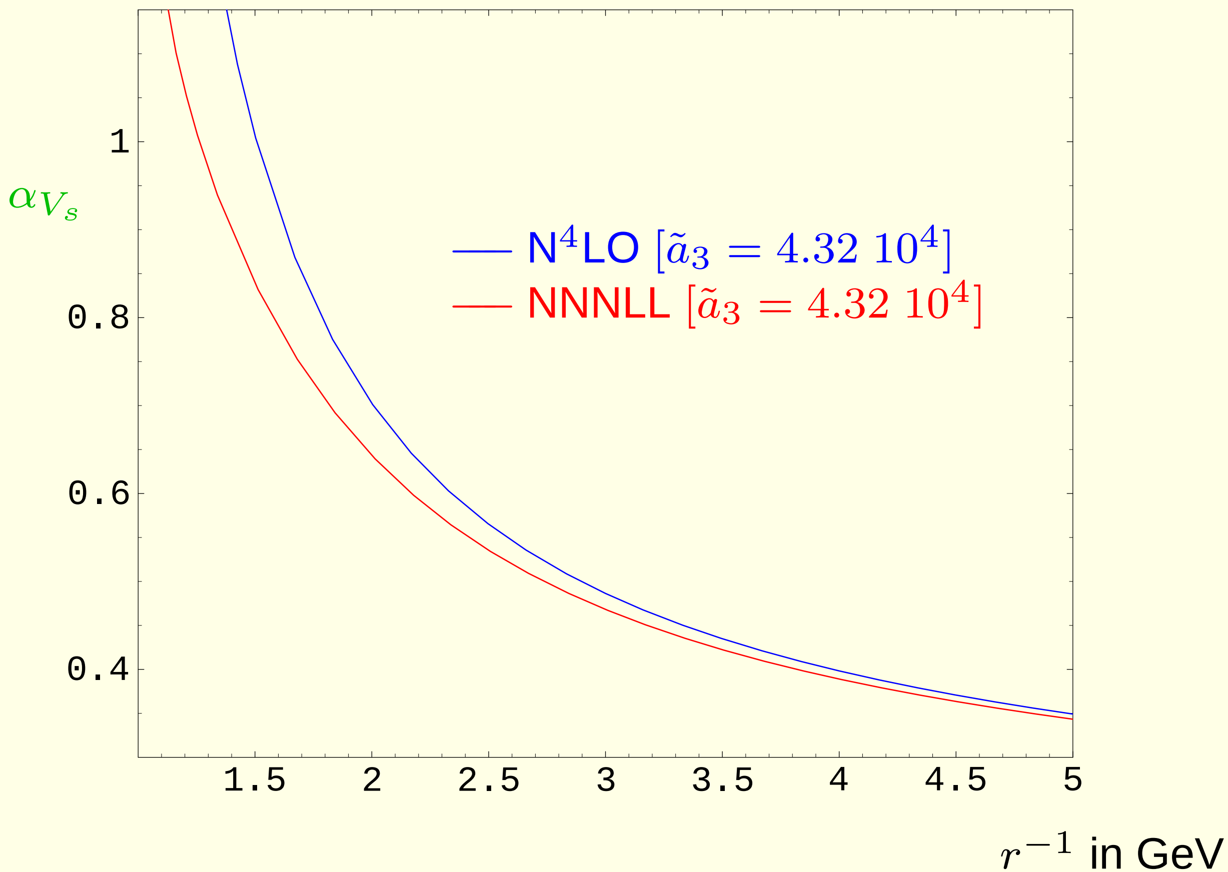
Static singlet potential at NNNLL



$(\mu = \alpha_s(r)/r;\ n_f = 4)$

[preliminary]

Static singlet potential at NNNLL



$(\mu = \alpha_s(r)/r;\ n_f = 4)$

[preliminary]

Conclusions

Non-relativistic EFTs provide a rigorous definition of the potential between two heavy quarks.

- In the perturbative regime, we have calculated the NNNLL expression of the static potential (up to the three-loop non-logarithmic piece) .

- This may become a key ingredient for the precision calculation of several threshold observables, e.g. quarkonium masses and top-quark pair production cross section near threshold.