Discrete Resonator Model

Maxwell approach:
- DRM from eigenmode expansion
- example: Tesla cavity → cavity signals

Empiric approach:
- network models for (quasi) periodic cavities
- field flatness and cavity spectrum
- field flatness and loss-parameter
- transient detuning

summary/conclusions
DRM from eigenmode expansion

\[ \nabla \times \nabla \times \mathbf{E} + \mu \varepsilon \frac{\partial^2}{\partial t^2} \mathbf{E} = -\frac{\partial}{\partial t} \mu \mathbf{J} \]

\[ \nabla \times \nabla \times \mathbf{E}_\nu = \mu \varepsilon \omega^2 \mathbf{E}_\nu \]

\[ \mathbf{E}(\mathbf{r}, t) = \sum \alpha_\nu(t) \mathbf{E}_\nu(\mathbf{r}) \]

\[ \mu \varepsilon \sum \left( \omega^2_\nu + \frac{\partial^2}{\partial t^2} \right) \alpha_\nu(t) \mathbf{E}_\nu(\mathbf{r}) = -\frac{\partial}{\partial t} \mu \mathbf{J} \]

\[ \frac{1}{2} \int \varepsilon \mathbf{E}_\nu \mathbf{E}_\mu dV = W_\nu \delta_{\nu\mu} \]

\[ \Rightarrow \left( \omega^2_\nu + \frac{\partial^2}{\partial t^2} \right) \alpha_\nu(t) = -\frac{\partial}{\partial t} \frac{1}{2} \int \mathbf{E}_\nu \mathbf{J} dV \]

\[ \mathbf{J} = q \dot{\mathbf{r}}_p(t) \delta(\mathbf{r} - \mathbf{r}_p(t)) \rightarrow g_\nu(t) = \frac{1}{2} q \dot{\mathbf{r}}_p(t) \cdot \mathbf{E}_\nu(\mathbf{r}_p(t)) \]

\[ u(t) \]

\[ C \quad \omega_0 \quad i(t) \]

\[ \Rightarrow \left( \omega^2_0 + \frac{d^2}{dt^2} \right) u(t) = -\frac{1}{C} \frac{d}{dt} i(t) \]

\[ u(t) = -\frac{1}{C} \int_0^t i(\tau) \cos(\omega_0(t-\tau)) d\tau \]
**EoM + eigenmode approach**

\[
\frac{d}{dt} \mathbf{r}_\mu = \mathbf{v}(p_\mu) \\
\frac{d}{dt} p_\mu = q \left\{ \mathbf{E}(r_\mu, t) + \mathbf{v}(p_\mu) \times \mathbf{B}(r_\mu, t) \right\} \\
\frac{d}{dt} \left( \begin{array}{c} \alpha_v \\ \beta_v \end{array} \right) = \left( \begin{array}{cc} 0 & \omega_v \\ -\omega_v & 0 \end{array} \right) \left( \begin{array}{c} \alpha_v \\ \beta_v \end{array} \right) - \left( \begin{array}{c} g_v \\ h_v \end{array} \right)_{\text{port}}
\]

with

\[
\mathbf{E}(t) = \sum \alpha_v(t) \mathbf{E}_v(r) \\
\mathbf{B}(t) = \sum \beta_v(t) \mathbf{B}_v(r)
\]

\[
\mathbf{B}_v = \frac{1}{\omega_v} \nabla \times \mathbf{E}_v \text{ for } \omega_v \neq 0
\]

\[
g_v(t) = \frac{1}{2W_v^{(m)}} \int \mathbf{E}_v J dV \\
J = q \sum v_\mu \delta(r - r_\mu)_{\text{beam}}
\]

\[
h_v(t) \sim (a-b) \int \mathbf{E}_v \mathbf{E}_{\perp, \text{port}} dA_{\text{port}}
\]

\[
a + b \sim \sum_v \alpha_v \int \mathbf{E}_v \mathbf{E}_{\perp, \text{port}} dA_{\text{port}} \text{ port coupling}
\]
port modes in matrix formalism

\[
\frac{d}{dt}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 & D_\omega \\ -D_\omega & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} - \begin{pmatrix} g + V(a-b) \\ 0 \end{pmatrix}
\]

beam \quad g

port \quad a = \text{all forward waves}

b = \text{all backward waves}

modes \quad \alpha = \text{all } E \text{ mode-amplitudes}

\beta = \text{all } B \text{ mode-amplitudes}

all coefficients can be calculated from EM eigenmode results

(but MWS does not support it)

the lossy eigenmode problem: \quad a = 0, \quad g = 0

\[
\downarrow
\]

\[
\frac{d}{dt}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -2VV'D_{wm} & D_\omega \\ -D_\omega & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}
\]

but it does not work too well if one considers not enough modes

example: TESLA cavity with modes below 2 GHz
example: TESLA cavity

\[ X_{\text{port}} = -193 \]
\[ \Delta x_{\text{pen}} = 8 \]

**direct calculation (modes below 2GHz):**

<table>
<thead>
<tr>
<th>f/GHz</th>
<th>Q/1E6</th>
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<tbody>
<tr>
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<tr>
<td>1.29991</td>
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**improved calculation**:  

<table>
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<td>2.57894</td>
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</tbody>
</table>

* compare:  
phase of reflection coefficient:

- **1.5 GHz**

- **3 MHz**

- **6 kHz**
the improved method

frequency domain (quantities with tilde)

\[\begin{align*}
  j\omega \begin{pmatrix} \tilde{a} \\ \tilde{\beta} \end{pmatrix} &= \begin{pmatrix} 0 & D_\omega \\ -D_\omega & 0 \end{pmatrix} \begin{pmatrix} \tilde{a} \\ \tilde{\beta} \end{pmatrix} - \begin{pmatrix} V (\tilde{a} - \tilde{b}) \\ 0 \end{pmatrix} \\
  \tilde{a} + \tilde{b} &= -2V'D_{wm} \tilde{a}
\end{align*}\]

\[\begin{align*}
  \downarrow \\
  \tilde{a} + \tilde{b} &= \tilde{Z}(\omega)(\tilde{a} - \tilde{b}) \quad \text{with} \quad \tilde{Z}(\omega) = \sum \frac{2j\omega}{\omega_v^2 - \omega^2} W_v v'_v v_v
\end{align*}\]

\[\begin{align*}
  \tilde{Z}(j\omega) &= \left( \sum_{v=1}^{N} \frac{2j\omega}{\omega_v^2 - \omega^2} W_v v'_v v_v \right) + \left( \sum_{v>N} \frac{2j\omega}{\omega_v^2 - \omega^2} W_v v'_v v_v \right) \\
  &= \tilde{Z}_k(j\omega) + \tilde{Z}_u(j\omega)
\end{align*}\]

this part is smooth in the frequency range of the known part; with some additional information one can find a good approximation
example: TESLA cavity

<table>
<thead>
<tr>
<th>f(PMC)/GHz</th>
<th>f(PEC)/GHz</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<tr>
<td>0.8612027</td>
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</tr>
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</table>

\[ \tilde{Z}_k (j \omega_{PMC}) = \infty \]
\[ \tilde{Z}_k (j \omega_{PEC}) + \tilde{Z}_u (\omega_{PEC}) = 0 \]

one-port system: \( Z_u \) can be calculated for PEC resonance frequencies
phase of reflection coefficient (again):
excitation of the modes* 1 to 11 by an ultra-relativistic point particle (q=1C)

these are the first 11 modes without divergence (\(\text{div } E\)) in vacuum. Modes 3-11 are related to the first band of monopole modes.

\[ E_z(z, r = 0) \text{ V/m} \]
port stimulation

\[ \tau_{\text{fill}} = \tau_{\pi \text{ mode}} \ln 2 \]

**-a vs time**

**b vs time**

this is time domain, \( \sim 5 \times 10^6 \) periods stimulation \( \omega_g \) on resonance of “pi-mode”

**blue curves** are sampled at \( t = nT_g \)
they can be interpreted as real part

**red curves** are sampled at \( t = nT_g + T_g/4 \)
they can be interpreted as real part
deviation from flat top

\[ \Delta E_z/E_{ref} \text{ vs time} \]

- **Cell 1**
  - \( \Delta E_z(nT_g) \)
  - \( \Delta E_z(nT_g + T_g/4) \)

- **Cell 9**
  - \( \Delta E_z(nT_g) \)
  - \( \Delta E_z(nT_g + T_g/4) \)

\[ \Delta E_z/E_{ref} \text{ vs time} \]

- **Ez at pick up position**
  - \( \Delta E_z(nT_g) \)
  - \( \Delta E_z(nT_g + T_g/4) \)

- **Ez at pickup**
  - **(cell 1 - cell 2 + ... + cell 9)/9**
  - \( \Delta E_z(nT_g) \)
  - \( \Delta E_z(nT_g + T_g/4) \)
“pickup” signal

spectrum of “pickup” signal and stimulation
“pickup” signal after LP filter

Filter\( \frac{E_{z\text{pick up}}}{E_{\text{ref}}} \) [V/m]

spectrum of “pickup” signal and LP filter

\(|A(\omega)|\max(|A(\omega)|)\) and Filter
"pickup" signal after BP filter

spectrum of "pickup" signal and BP filter
empiric approach:

network models for (quasi) periodic cavities

geometry

1 resonance, 2x1 ports, electric coupling

1 resonance, 2x1 ports, magnetic coupling

2 resonances, 2x2 ports, magnetic coupling

K. Bane, R. Gluckstern: The Transverse Wakefield of a Detuned X-Band Accelerator Structure, SLAC-PUB-5783, March 1992
approach from network theory

b1, b2: boundary blocks with $n$ respectively $n+1$ ports

c1, c2, ... c9: cell blocks with $2n$ ports

simplifications: c2, ... c8 (or c1, ... c9) are identical and symmetric
use periodic solutions of 3d system to characterize cell blocks
special treatment of boundary blocks
system with beam

beam ports with delay are connected in series

\[ i_a(t) \rightarrow u_a(t) \]

- \[ i_b(t) = i_a(t - \tau) \]
- \[ u_b(t) = u_a(t - \tau) \]
the general symmetric 2n-port network

\[
\begin{pmatrix}
\tilde{u}_1(j\omega) \\
\tilde{u}_2(j\omega) \\
\vdots \\
\tilde{u}_{2n}(j\omega)
\end{pmatrix} = \tilde{Z}(j\omega)
\begin{pmatrix}
\tilde{i}_1(j\omega) \\
\tilde{i}_2(j\omega) \\
\vdots \\
\tilde{i}_{2n}(j\omega)
\end{pmatrix}
\]

with

\[
\tilde{Z}(j\omega) = \sum_{v=1}^{M} \frac{j\omega}{\omega_v^2 - \omega^2} A_v + j\omega A_\infty
\]

and

\[
A_v = \begin{pmatrix}
\mathbf{r}_v' \\
\mathbf{s}_v'
\end{pmatrix}
\begin{pmatrix}
\mathbf{r}_v \\
\mathbf{s}_v
\end{pmatrix} + \begin{pmatrix}
\mathbf{s}_v' \\
\mathbf{r}_v'
\end{pmatrix}
\begin{pmatrix}
\mathbf{r}_v \\
\mathbf{s}_v
\end{pmatrix}
\]

\[
A_\infty = \cdots
\]
Relative spectral deviation

\[ \Delta_i = \frac{f_i}{F_i} - \frac{f_9}{F_9} \]

RSD is a measure for the change of the mode spectrum of one cavity compared to itself.
field flatness and mode spectrum

Is there a direct relation between the field flatness of the accelerating mode and the resonance frequencies of modes in the same band?

test it for a simpler problem:

even simpler:

no: both setups have the same eigen-frequencies, but different flatness/eigenvectors

\[
C_1 = C_2 = 1.25, \quad C_c = 5
\]

\[
\Rightarrow \vec{i}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{i}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \omega_1 = \sqrt{4/5}
\]

\[
C_1 = \frac{10}{11 - \sqrt{3}}, \quad C_2 = \frac{10}{9 - \sqrt{3}}, \quad C_c = 10/\sqrt{3}
\]

\[
\Rightarrow \vec{i}_1 = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}, \quad \vec{i}_2 = \frac{1}{2} \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix} \quad \omega_2 = \sqrt{6/5}
\]
field flatness and loss parameter

again the discrete network:

\[
\begin{align*}
C_{i} & \quad \frac{1}{2} \quad \frac{1}{2} \\
\nu & \quad \nu & \quad \nu & \quad \nu \\
\omega & \quad \omega & \quad \omega & \quad \omega \\
\pi & \quad \pi & \quad \pi & \quad \pi
\end{align*}
\]

loss-parameter:

\[
k = \frac{|V|^2}{4W_{tot}} \approx \frac{1}{2\omega_0^2} \sum \left( \frac{1}{C_\nu} \right) \frac{1}{L_\nu^2} \approx \frac{1}{2\omega_\pi^2} \sum \left( \frac{1}{C_\nu} \right) \tilde{i}_\nu^2
\]

\[
\tilde{i}_\nu \sim (1 + x_\nu) \quad \text{deviation from flat field}
\]

\[
k \sim \frac{\sum (1 + x_\nu)^2}{9 \sum (1 + x_\nu)^2} \approx \frac{1}{1 + \langle x_\nu^2 \rangle}
\]

if \( \langle x_\nu \rangle \approx 0 \)

for \( \sigma_x = 0.1 \) the loss parameter is reduced by only 1%!
simulation for network with tolerances

assumption: error tolerances of network parameters, uncorrelated ...
correlation between field flatness and relative spectral deviation

assumption: error tolerances of network parameters, uncorrelated ...
time dependent detuning

\[
\frac{d}{dt} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 & D_\omega \\ -D_\omega & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} - \begin{pmatrix} V(a-b) \\ 0 \end{pmatrix}
\]

\[a + b = -2V^t \alpha\]

\[
\frac{d}{dt} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 & D_\omega(t) \\ -D_\omega(t) & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} - \begin{pmatrix} V(t)(a-b) \\ 0 \end{pmatrix}
\]

\[a + b = -2V^t(t) \alpha\]

for instance no excitation \((a = b)\) →

\[
\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 & \int_0^t D_\omega(\tau) d\tau \\ -\int_0^t D_\omega(\tau) d\tau & 0 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix}
\]

This would mean all the modes are independently ringing, there is no mode conversion.
example:

\[
\frac{d}{dt} z = \begin{pmatrix} 0 & \mathbf{R}(t) \\ \mathbf{R}(t) & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbf{D}_\omega \\ -\mathbf{D}_\omega & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbf{R}(t) \\ \mathbf{R}(t) & 0 \end{pmatrix}^t + \mathbf{q} \quad \text{with} \quad \mathbf{D}_\omega = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}
\]

\[
\mathbf{R}(t) = \begin{pmatrix} \cos \varphi(t) & \sin \varphi(t) \\ -\sin \varphi(t) & \cos \varphi(t) \end{pmatrix}
\]

system matrix is anti symmetric $\rightarrow$ energy conservation with $W \sim \mathbf{z}' \mathbf{z}$

eigenvalues are time invariant $\lambda = \{ \pm j\omega_1, \pm j\omega_2 \}$

but eigenvectors are time dependent

next slide:

numerical calculation for $\omega_1 = 1.98\pi$, $\omega_2 = 2.02\pi$

the effect of mode conversion is weak if the time scale of $\varphi(t)$ is long compared to the resonances
no excitation
$q = 0$

initial state:
$\omega_1$ resonance is ringing

final state:
both resonances are ringing
summary/conclusion

Maxwell approach: field eigenmode expansion

- eigenmode expansion is effective to analyse cavity signals in the frequency range of the first monopole band → signals seen by couplers and pickups
- pickups are more sensitive to non-accelerating monopole modes than the beam
- eigenmode expansion is standard for long range effects

Empiric approach: discrete network

- discrete network models allow qualitative insight
- it is easy to analyze discrete models and to consider random effects
- it is difficult to relate network parameters to geometric properties and imperfections; it is in principle possible
- loss-parameter is very insensitive to field flatness; but the peak field is sensitive!
- no sharp correlation between relative spectral deviation and flatness
- it is possible to calculate time dependent resonance, but modeling requires (some) caution