Theory of Elementary Particles

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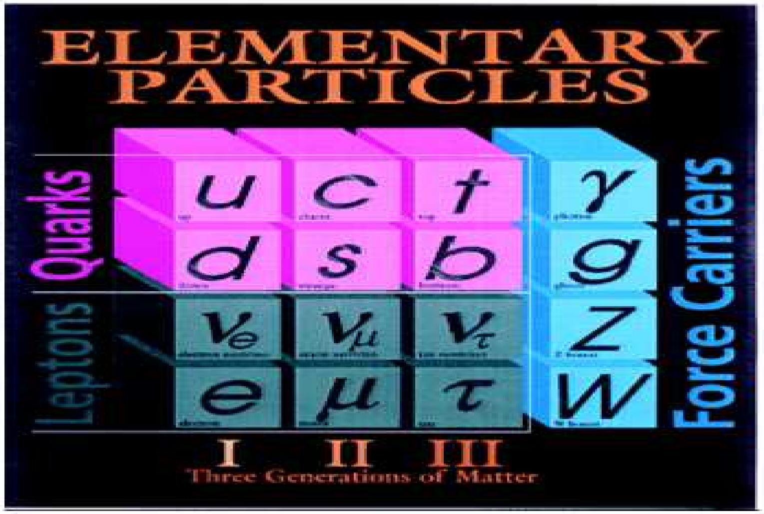
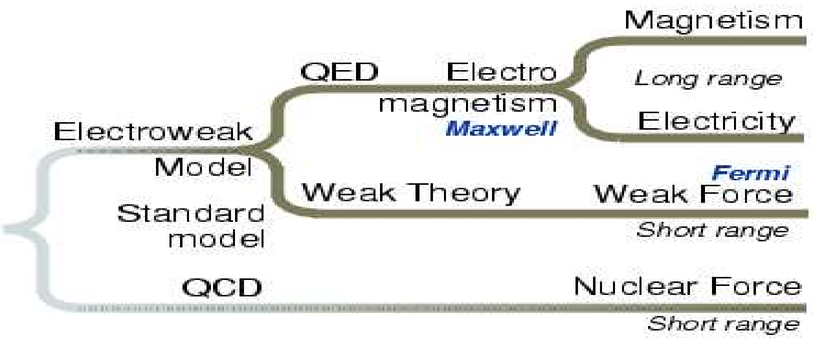
The Standard Model





Theories:
RELATIVISTIC/QUANTUM

Structure of the first generation of leptons and quarks in SM

$$\begin{pmatrix} u & u & u \\ d & d & d \end{pmatrix}^{L}_{1/6}$$

$$\begin{pmatrix} \nu \\ e \end{pmatrix}^{L}_{-1/2}$$

$$(u \;\; u \;\; u)^{R}_{2/3}$$

$$(d \;\; d \;\; d)^{R}_{-1/3}$$

$$(e)^{R}_{-1}$$

No ν^{R}

SU(3) x SU(2) x U(1)

mixed, not unified

- The hypercharge quantum numbers Y are fixed by the relation $Q = I_3 + Y$
- Q are the electric charges (in units of the proton charge):

$$Q\begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} +2/3 \\ -1/3 \end{pmatrix}; Q\begin{pmatrix} \nu_e \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

- I_3 is the third-component of the weak isospin:

$$I_3\begin{pmatrix} u \\ d \end{pmatrix}_{\mathrm{L}} = \begin{pmatrix} +1/2 \\ -1/2 \end{pmatrix}; I_3\begin{pmatrix} \nu_e \\ e \end{pmatrix}_{\mathrm{L}} = \begin{pmatrix} +1/2 \\ -1/2 \end{pmatrix}$$

- $I = I_3 = 0$ for $u_{\mathrm{R}}, d_{\mathrm{R}}, e_{\mathrm{R}}$

Plan of these lectures

- Introduction to Quantum Field Theory (QFT)
- Gauge Invariance and Quantum Electrodynamics (QED)
- An elementary QED process $e^+e^- \longrightarrow \mu^+\mu^-$
- Gauge Invariance and Quantum Chromodynamics (QCD)
- The QCD running coupling constant $\alpha_s(Q^2)$
- Applications of perturbative QCD to hard processes
- The Electroweak Theory
- The Higgs Mechanism
- CKM Matrix and CP Violation
- Precision Tests of the EW theory
- Higgs Boson Physics
- Gauge coupling unification
- Conclusions

Recommended References

Books

- Relativistic Quantum Mechanics & Relativistic Quantum Fields
  James D. Bjorken, Sidney D. Drell, McGraw-Hill Book Company (1965)

- An Introduction to Quantum Field Theory
  Michael E. Peskin, Daniel V. Schroeder, ABP, Westview Press (1995)

- Modern Quantum Field Theory - A Concise Introduction
  Thomas Banks, Cambridge University Press (2008)

- Foundations of Quantum Chromodynamics
  Taizo Muta, World Scientific -revised edition (2005)

- Dynamics of the Standard Model
  John F. Donoghue, Eugene Golowich, Barry R. Holstein, Cambridge University Press (Revised version 1994)

Review Articles

- A QCD Primer
  G. Altarelli, arXiv Preprint hep-ph/0204179
- The Standard Model of Electroweak Interactions
  A. Pich, arXiv Preprint 0705.4264 [hep-ph]
- Concepts of Electroweak Symmetry Breaking and Higgs Physics
  M. Gomez-Bock, M. Mondragon, M. Mühlleitner, M. Spira, P.M. Zerwas
  arXiv Preprint 0712.2419 [hep-ph]
- Essentials of the Muon $g-2$
  F. Jegerlehner, DESY Report 07-033, arXiv:hep-ph/0703125 (2007)
- Review papers on the Standard Model Physics
  Particle Data Group (URL:http://pdg.lbl.gov/)
  - I. Hinchliffe (Quantum Chromodynamics)
  - J. Erler & P. Langacker (Electroweak Model and Constraints on New Physics)
  - A. Ceccucci, Z. Ligeti, Y. Sakai (The CKM Quark-Mixing Matrix)
  - G. Bernardi, M. Carena, T. Junk (Higgs Bosons: Theory and Searches)
  - B. Foster, A.D. Martin, M.G. Vincter (Structure Functions)

Quantum Field Theory - Introduction

- Physical systems can be characterized by essentially two features, roughly speaking, size (large or small) and speed (slow or fast)

- Velocity $\boldsymbol{v} \ll \boldsymbol{c}$: For slowly moving objects (compared to the speed of light), we have the Classical (Newtonian) mechanics to describe this motion

- Velocity $\boldsymbol{v} \longrightarrow \boldsymbol{c}$: For fast moving system (approaching the speed of light), we have Relativistic (Einstein's) mechanics

- To describe objects at small distances (say, atoms), energy and other attributes (such as angular momenta) are quantized in units of the Planck's constant $\hbar \neq \mathbf{0}$. For quantum systems $\hbar \neq \mathbf{0}$ moving with non-relativistic velocities $\boldsymbol{v} \ll \boldsymbol{c}$, we move from Classical mechanics to Non-relativistic Quantum mechanics

- For $\hbar \neq \mathbf{0}$ and $\boldsymbol{v} \longrightarrow \boldsymbol{c}$, we have Relativistic Quantum mechanics (RQM)

- RQM is insufficient as it does not account for the particle production and annihilation. However, $\underline{\boldsymbol{E} = \boldsymbol{m}\boldsymbol{c}^{\mathbf{2}}}$ allows pair creation

- Quantum Mechanics + Relativity + Particle creation and annihilation $\Longrightarrow$ Quantum Field Theory

<u>Quantum Field Theory</u>

- Application of Quantum mechanics to dynamical systems of fields $\boldsymbol{\Phi(t, \vec{x})}$

Preliminaries

Space-time coordinates & Lorentz scalars

- Natural units $(c = \hbar = 1)$. In these units, the Compton wavelength of the electron is $1/m_e \simeq 3.86 \times 10^{-11}$ cm and the electron mass is $m_e \simeq 0.511$ MeV. In the same units, $m_p \simeq 938$ MeV and $m_t \simeq 172$ GeV
- Space-time coordinates (t, x, y, z) are denoted by the *contravariant* four vector

$$x^\mu = (x^0, x^1, x^2, x^3) \equiv (t, x, y, z) = (t, \vec{r})$$

- The *covariant* four-vector x_μ is obtained by changing the sign of the space components

$$x_\mu = (x_0, x_1, x_2, x_3) = g_{\mu\nu} x^\nu \equiv (t, -x, -y, -z) = (t, -\vec{r})$$

with the metric $g_{\mu\nu} = g^{\mu\nu}$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- The inner product, yielding a Lorentz-scalar, is $x^2 = t^2 - \vec{r}^2 = t^2 - x^2 - y^2 - z^2$
- Momentum vectors are defined as:

$$p^\mu = (E, p_x, p_y, p_z)$$

and the inner product (Lorentz scalar) is:

$$p_1.p_2 = p_1^\mu p_{2\mu} = E_1 E_2 - \vec{p}_1.\vec{p}_2 \quad \& \quad x.p = tE - \vec{r}.\vec{p}$$

Preliminaries

Momentum Operator in Coordinate representation

$$p^\mu = i\frac{\partial}{\partial x_\mu} \equiv \left(i\frac{\partial}{\partial t}, \frac{1}{i}\vec{\nabla}\right) \equiv i\partial^\mu$$

- This transforms as a *contravariant* four-vector

$$p^\mu p_\mu = -\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu} = \vec{\nabla}^2 - (\frac{\partial}{\partial t})^2 = -\Box$$

- The four-vector potential of the electromagnetic field is defined as

$$A^\mu = (\Phi, \vec{A}) = g^{\mu\nu}A_\nu$$

where $\Phi = \Phi(t, \vec{x})$ is Scalar potential; $\vec{A} = \vec{A}(t, \vec{x})$ is Vector potential

- Maxwell Equations: $\partial_\mu F^{\mu\nu} = eJ^\nu$
where e is the electric charge ; Sommerfeld Constant $\alpha = \frac{e^2}{4\pi} \sim 1/137$ and J^ν is the electromagnetic current
- The field strength tensor is defined as $F^{\mu\nu} = \frac{\partial}{\partial x_\nu}A^\mu - \frac{\partial}{\partial x_\mu}A^\nu$, and

$$\begin{aligned}\vec{E} &= (F^{01}, F^{02}, F^{03})\\ \vec{B} &= (F^{23}, F^{31}, F^{12})\end{aligned}$$

Elements of Classical Field Theory

Action S and Lagrangian L

- Action $S = \int_{t_1}^{t_2} L dt$: Time integral of the Langrangian L
- Lagrangian $L = T - V$ (Kinetic Energy - Potential Energy); $L = L(q(t), \dot{q}(t))$

- In classical mechanics, the Lagrange function is constructed from the generalized coordinates $q(t)$ and velcocities $\dot{q}(t)$
- Hamilton's Principle of Least Action: Dynamics of the particle traversing a path $q(t)$ is determined from the condition $\delta S = \delta \int_{t_1}^{t_2} L(q(t), \dot{q}(t)) dt = 0$
- Also in Quantum mechanics, Classical path minimises Action [Feynman; see Feynman Lectures, Vol. II]
- Lagrangian in field theory is constructed from $\Phi(t, \vec{x})$, $\dot{\Phi}(t, \vec{x})$, $\vec{\nabla}\Phi(t, \vec{x})$

$$L = \int d^3x \underbrace{\mathcal{L}\left(\Phi(t, \vec{x}), \dot{\Phi}(t, \vec{x}), \vec{\nabla}\Phi(t, \vec{x})\right)}$$

- $\mathcal{L}$ is the Lagrangian density; $S = \int L dt = \int d^4x \mathcal{L}\left(\Phi(t, \vec{x}), \dot{\Phi}(t, \vec{x}), \vec{\nabla}\Phi(t, \vec{x})\right)$
- Minimising the Action, i.e., $\delta S = 0$ leads to the Euler-Lagrange Equation of motion

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\Phi}} + \vec{\nabla}\frac{\partial \mathcal{L}}{\partial(\vec{\nabla}\phi)} = 0$$

- Hamiltonian: $H = \int d^3x \left[\pi(x)\dot{\phi}(x) - \mathcal{L}\right] = \int d^3x \mathcal{H}$

Constructing $\mathcal{L}$: The free field case

Spin 0 field: $\Phi(t, \vec{x})$

- For a free scalar field with mass m, the Lagrangian density is

$$\mathcal{L}^{\Phi}_{\text{free}} = \frac{1}{2}\left((\frac{\partial}{\partial t}\Phi)^2 - (\vec{\nabla}\Phi)^2\right) - \frac{1}{2}m^2\Phi^2$$

- The resulting field equation is called the Klein-Gordon equation

$$(\Box + m^2)\Phi(t, \vec{x}) = 0 \quad \text{with} \quad \Box = (\frac{\partial}{\partial t})^2 - \vec{\nabla}^2$$

Spin $\frac{1}{2}$ field: $\Psi(t, \vec{x})$

- $\Psi(t, \vec{x}) = \Psi_\alpha(t, \vec{x})$ is a 4-component spinor $(\psi_1, \psi_2, \psi_3, \psi_4)$
- Lagrangian density of a free Dirac (=spin $\frac{1}{2}$) field with mass m

$$\mathcal{L}^{\psi}_{\text{free}} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

- γ^μ are 4×4 matrices. The resulting field equation is called the Dirac equation

$$(i\partial\!\!\!/ - m)\psi(t, \vec{x}) = 0$$

Spin 1 field with $m = 0$: (Example: Electromagnetic field) $A^\mu(t, \vec{x})$

- Lagrangian density of a free spin-1 field: $\mathcal{L}^{A}_{\text{free}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$
- Yields Maxwell equations in the absence of charge and current densities

$$\partial_\mu F^{\mu\nu}(t, \vec{x}) = 0$$

Noether's Theorem

Relationship between Symmetries & Conservation Laws in CFT

- Noether's theorem concerns continuous transformations on the field $\phi(x)$, which in the infinitesimal form can be written as
$$\phi(x) \longrightarrow \phi'(x) = \phi(x) + \alpha\Delta\phi(x)$$
- This is a symmetry transformation if it leaves the equation of motion (EOM) invariant, which is ensured if the action S is invariant
- More generally, one can allow S to change by a surface term, since such a term leaves the Euler-Lagrange EOM invariant
- Thus, for some $\mathcal{J}^\mu(x)$, the Lagrangian must be invariant up to a 4-divergence
$$\mathcal{L}(x) \longrightarrow \mathcal{L}(x) + \alpha\partial_\mu\mathcal{J}^\mu(x)$$
- Let us compute explicitly $\alpha\Delta\mathcal{L}$, and the result is
$$\alpha\Delta\mathcal{L} = \alpha\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) + \alpha\left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right]\Delta\phi$$
- The second term vanishes due to the EOM. Setting the first term equal to $\alpha\partial_\mu\mathcal{J}^\mu(x)$
$$\partial_\mu J^\mu(x) = 0 \text{ for } J^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - \mathcal{J}^\mu(x)$$
- This result states that the current $J^\mu(x)$ is conserved
- Noether's theorem can also be stated in the form that the charge
$$Q \equiv \int_{\text{all space}} J^0(x)d^3x$$
  is constant in time

Emmy Noether (1882 - 1935)

Examples of conserved currents and charges

- Example 1: Lagrangian of a complex scalar field with mass m

$$\mathcal{L}(\phi) = |\partial_\mu \phi|^2 - m^2 |\phi|^2$$

• $\mathcal{L}(\phi)$ is invariant under $\phi(x) \longrightarrow e^{i\alpha}\phi(x)$. Under this transformation:

$$\alpha\Delta\phi(x) \rightarrow i\alpha\phi(x); \;\; \alpha\Delta\phi^*(x) \rightarrow -i\alpha\phi^*(x)$$

• Conserved Noether currents:

$$J_\mu = i[(\partial_\mu \phi^*)\phi - \phi^*(\partial_\mu \phi)$$

• Adding a term $eJ_\mu(x)A^\mu(x)$ to $\mathcal{L}(\phi)$, where $A^\mu(x)$ is the electromagnetic field, we interpret $eJ_\mu(x)$ as the electromagnetic current density with $\int d^3x J_0(x)$ as the conserved electric charge

- Example 2: Noether's theorem applied to space-time transformations
Let us describe the infinitesimal transformation as $x^\mu \longrightarrow x^\mu - a^\mu$

• Field Transformation:$\phi(x) \longrightarrow \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x)$

• The Lagrangian is also a scalar, so it must transform in the same way

$$\mathcal{L} \longrightarrow \mathcal{L}' = \mathcal{L} + a^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L})$$

• Noether's theorem predicts four conserved currents: The *stress-energy tensor*, also called the *energy-momentum tensor* of the field $\phi(x)$

$$T^\mu_\nu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L}\delta^\mu_\nu$$

• The conserved charge associated with time translation is the Hamiltonian

$$H = \int T^{00}(x) d^3x = \int \mathcal{H} d^3x \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right]$$

• The conserved charge associated with spatial translations are the momentum

$$P^i = \int T^{0i} d^3x = -\int \pi \partial_i \phi d^3x$$

Field Quantization

- In QM, the coordinate $q(t)$ and the conjugate coordinate $p(t) = \frac{\partial L}{\partial \dot{q}(t)}$ satisfy the commutation relation (recall, $\hbar = c = 1$):

$$[q(t), p(t)] = i \quad \textbf{where} \quad [A, B] = A.B - B.A$$

- In QFT, the cordinate $q(t)$ is replaced by the corresponding field and its conjugate field. For the Spin 0 field: $\Phi(t, \vec{x})$, the conjugate field is:

$$\pi(t, \vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}(t, \vec{x})}$$

Scalar boson field quantization

$$\left[\Phi(t, \vec{x}), \pi(t, \vec{x}')\right] = i\delta^3(\vec{x} - \vec{x}')$$

- For the Spin $\frac{1}{2}$ field: $\Psi_\alpha(t, \vec{x})$, the conjugate field is: $\pi_\alpha(t, \vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\Psi}_\alpha(t, \vec{x})}$
Dirac fermion field quantization

$$\left\{\Psi_\alpha(t, \vec{x}), \pi_\beta(t, \vec{x}')\right\} = i\delta^3(\vec{x} - \vec{x}')\delta_{\alpha\beta}$$

where $\{A, B\} = A.B + B.A$ is the anti-commutator; the use of anti-commutator for fermion fields instead of commutator for the boson fields is due to the spin $1/2$ nature of the fermions

- For the Spin 1 field with $m = 0$, $A^{\mu}(t, \vec{x})$ $(\mu = 0, 1, 2, 3)$, the conjugate field is:

$$\pi^{\mu}(t, \vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{A}_{\mu}(t, \vec{x})}$$

Spin-1 $(m = 0)$ field quantization

$$\left[A^{\mu}(t, \vec{x}), \pi^{\nu}(t, \vec{x}')\right] = g^{\mu\nu} i \delta^3(\vec{x} - \vec{x}')$$

QFT & particle interpretation

- Free field equation for a Scalar field with mass m: $(\Box + m^2)\Phi(t, \vec{x}) = 0$
  General solution:

$$\Phi(t, \vec{x}) \propto \int dE\, d^3p \delta(E^2 - \vec{p}^2 - m^2) \left(a(E, \vec{p}) e^{-i(Et - \vec{p}.\vec{x})} + a^{\dagger}(E, \vec{p}) e^{+i(Et - \vec{p}.\vec{x})}\right)$$

  - $\mathrm{a}(p) = \mathrm{a}(E, \vec{p})$ is the field operator, $\mathrm{a}^{\dagger}(p) = \mathrm{a}^{\dagger}(E, \vec{p})$ is the hermitian conjugate field operator

- Field quantization: $[\Phi(t, \vec{x}), \pi(t, \vec{x}')] = i\delta^3(\vec{x} - \vec{x}') \implies$ (with $p \equiv (E, \vec{p})$):

$$\left[\mathrm{a}(p), \mathrm{a}^{\dagger}(p')\right] = 2E\delta^{(3)}(\vec{p} - \vec{p}'); \quad \left[\mathrm{a}(p), \mathrm{a}(p')\right] = 0; \quad \left[\mathrm{a}^{\dagger}(p), \mathrm{a}^{\dagger}(p')\right] = 0$$

Fock space of particles

Hamilton Operator

$$\mathbf{H} = \int d^3x(\pi\dot{\Phi} - \mathcal{L}) = \int dE\, d^3p\, \delta(E^2 - \vec{p}^2 - m^2)\mathbf{E}\ \mathbf{a}^\dagger(p)\mathbf{a}(p)$$

- $\mathbf{N(p)} \equiv \mathbf{a}^\dagger(p)\mathbf{a}(p)$ is the Number Operator

$$N(p)|n(p)\rangle = \mathbf{n}(p)|n(p)\rangle$$

- $n(p)$ is the number of particles with spin 0, mass m, having an energy between E and $E + dE$ and momentum between $\vec{p}$ and $\vec{p} + d\vec{p}$, $E = +\sqrt{\vec{p}^2 + m^2}$
- The particle creation $\mathbf{a}^\dagger(p)$ and annihilation $\mathbf{a}(p)$ operators act as follows:

$$\begin{aligned} \mathbf{a}^\dagger(p)|n(p)\rangle &= \sqrt{n(p)+1}|(n+1)(p)\rangle \\ \mathbf{a}(p)|n(p)\rangle &= \sqrt{n(p)}|(n-1)(p)\rangle \end{aligned}$$

- They provide the basis for particle production and annihilation in QFT
- Energy of the ground state is normalized to zero: $H|0\rangle = 0$
- Multiparticle states obeying Bose-Einstein Statistics are built as follows

$$|n_1(p_1), ..., n_m(p_m)\rangle \propto (a^\dagger(p_1))^{n_1} ... (a^\dagger(p_m))^{n_m}|0\rangle$$

QFT Roadmap

- Quantization of a classical theory starts with the Field Equations
- Knowing these, we seek a Lagrangian which, via Hamilton's principle, reproduces them
- Having the Lagrangian, it is then possible to identify the canonical momenta and their conjugates, and carry out the quantization procedure, in accordance with the following commutation relations:

$$\begin{aligned} [p_i(0), q_j(0)] &= -i\delta_{ij} \\ [p_i(0), p_j(0)] &= 0 \\ [q_i(0), q_j(0)] &= 0 \end{aligned}$$

- It is understood from this analogy, we expect $\Phi(t, \vec{x})$ to play the role of the coordinate $q_i(t)$ and $\partial\Phi(t, \vec{x})$ to correspond to $\dot{q}_i(t)$; the discrete label i is replaced by the continuous coordinate variable $\vec{x}$, which in the Heisenberg picture (used for quantization) becomes a function of both space and time
- With this step, the fields $\Phi(t, \vec{x})$ and $\Pi(t, \vec{x})$ become operators in a Hilbert space, operating upon the state vectors
- Physical states form a complete set of states in the Hilbert space
- For the Dirac particles, the commutatation relations are replaced by the anticommutation relations

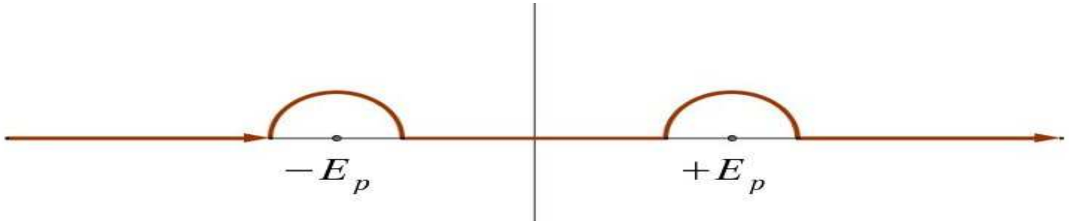
The Klein-Gordon Propagator

- In the Heisenberg picture, easier to discuss time-dependent quantities, the operators $\pi(\mathrm{x})$ and $\phi(\mathrm{x})$ are made time-dependent (likewise for $\pi(x) = \pi(\mathrm{x}, t)$)
$$\phi(x) = \phi(\mathrm{x}, t) = e^{iHt}\phi(\mathrm{x})e^{-iHt}$$
- Heisenberg EOM: $i\frac{\partial}{\partial t}\mathcal{O} = [\mathcal{O}, H]$
- Computing the time-dependence of $\phi(x)$ and $\pi(x)$
$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_{\mathrm{p}}e^{-ip.x} + a_{\mathrm{p}}^{\dagger}e^{ip.x}\right)|_{p^0=E_p} \ ,$$
$$\pi(\mathrm{x}, t) = \frac{\partial}{\partial t}\phi(\mathrm{x}, t)$$
- Causality requirement: $[\phi(x), \phi(y)] = 0 \text{ for } (x-y)^2 < 0$
- The amplitude for a particle to propagate from a point x to a point y is:
$$\langle 0|\phi(x)\phi(y)|0\rangle$$
- Each operator ϕ is a sum of a and $a^{\dagger}$ operators, but only the term
$$\langle 0|a_{\mathrm{p}}a_{\mathrm{q}}^{\dagger}|0\rangle = (2\pi^3)\delta^{(3)}(\mathrm{p}-\mathrm{q}) \text{ survives}$$
$$\Longrightarrow \quad D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip.(x-y)}$$
- Hence, the commutator is: $[\phi(x), \phi(y)] = D(x-y) - D(y-x)$
- For $(x-y)^2 > 0$, it does not vanish
- For $(x-y)^2 < 0$, it vanishes due to a special Lorentz transformation, taking $(x-y) \to -(x-y)$. Thus, the requirement of causality is fulfilled:
$$[\phi(x), \phi(y)] = 0 \text{ for } (x-y)^2 < 0$$

The Klein-Gordon Propagator (Contd.)

- As $[\phi(x), \phi(y)]$ is a c-number, $[\phi(x), \phi(y)] = \langle 0|[\phi(x), \phi(y)]|0\rangle$
- Let us compute this quantity

$$\begin{aligned}
\langle 0|[\phi(x), \phi(y)]|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}(e^{-ip.(x-y)} - e^{+ip.(x-y)}) \\
&= \int \frac{d^3p}{(2\pi)^3}\left[\frac{1}{2E_p}e^{-ip.(x-y)}|_{p^0=E_p} + \frac{1}{-2E_p}e^{-ip.(x-y)}|_{p^0=-E_p}\right] \\
&\underbrace{=}_{x^0>y^0} \int \frac{d^3p}{(2\pi)^3}\int\frac{dp^0}{2\pi i}\frac{-1}{p^2-m^2}e^{-ip.(x-y)}
\end{aligned}$$

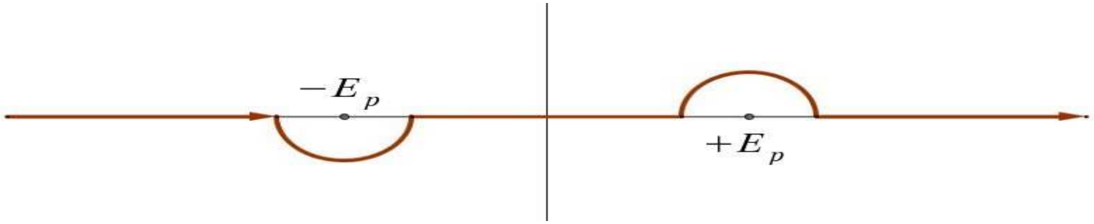


- For $x^0 > y^0$, the last line is obtained by closing the contour of integration below to pick up both the poles at $p^0 = E_p$ and $p^0 = -E_p$.
- For $x^0 < y^0$, the contour of integration can be closed above excluding both the poles, giving zero
- The last line above together with the prescription of going around the poles defines the Retarded Green's function: $D_R(x-y) \equiv \theta(x^0 - y^0)\langle 0|[\phi(x), \phi(y)]|0\rangle$

Feynman propagator for a Klein-Gordon particle

$$D_R(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip.(x-y)}$$

- One can compute to show that $D_R(x-y)$ is a Green's function of the Klein-Gordon operator, i.e., $(\partial^2 + m^2) D_R(x-y) = -i\delta^{(4)}(x-y)$
- The p^0-integral in $D_R(x-y)$ can be evaluated by selecting four different contours. A different presciption is due to Feynman



- *Feynman Prescription*: poles at $p^0 = \pm(E_p - i\epsilon)$, displaced above and below the real axis. A convenient way is to write it as

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip.(x-y)}$$

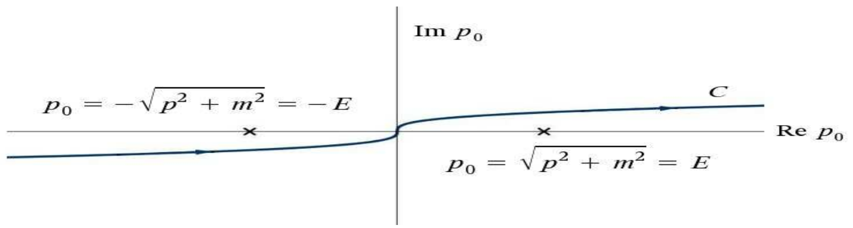
- For $(x^0 > y^0)$, we can perform the p^0 integral by closing the contour below, obtaining
$$D(x-y) = \langle 0 | [\phi(x), \phi(y)] | 0 \rangle$$
- For $(x^0 < y^0)$, we can perform the p^0 integral by closing the contour above, obtaining
$$D(y-x) = \langle 0 | [\phi(y), \phi(x)] | 0 \rangle$$

- Thus, we have

$$D_F(x-y) = \begin{cases} D(x-y) & \text{for } x^0 > y^0 \\ D(y-x) & \text{for } x^0 < y^0 \end{cases}$$
$$= \theta(x^0 - y^0)\langle 0|[\phi(x), \phi(y)]|0\rangle + \theta(y^0 - x^0)\langle 0|[\phi(y), \phi(x)]|0\rangle$$
$$= \langle 0|T\phi(x)\phi(y)|0\rangle$$

- The last line above defines the "time ordering" symbol T
- Applying $(\partial^2 + m^2)$ to the last Eq. above, one can verify $D_F(x-y)$ is the Green's function of the Klein-Gordon operator, and it is called the Feynman Propagartor for a Klein-Gordon particle
- Likewise, the Feynman Propagator for a free Dirac particle is

$$S_F(x-y) = \int \frac{d^4p}{2\pi^4} e^{-ip.(x-y)} \frac{\not{p} + m}{p^2 - m^2 + i\epsilon}$$



- $S_F(x-y)$ satisfies the Green's function equation

$$(i\nabla_x - m)S_F(x-y) = \delta^{(4)}(x-y)$$

- In the momentum space: $S_F(p) = \frac{1}{\not{p} - m + i\epsilon}$

Gauge Invariance & Quantum Electrodynamics (QED)

Lagrangian describing a free Dirac fermion

$$\mathcal{L}_0 = i\,\overline{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - m\,\overline{\psi}(x)\psi(x)$$

$\mathcal{L}_0$ is invariant under *global* $U(1)$ transformations

$$\psi(x) \quad \xrightarrow{\mathrm{U(1)}} \quad \psi'(x) \equiv \exp\{iQ\theta\}\,\psi(x)$$

$Q\theta$ an arbitrary real constant; the phase of $\psi(x)$ is a convention-dependent quantity without physical meaning

• However, $\mathcal{L}_0$ no longer invariant if the phase transformation depends on the space-time coordinate, i.e., under *local* phase redefinitions $\theta = \theta(x)$

$$\partial_\mu\psi(x) \quad \xrightarrow{\mathrm{U(1)}} \quad \exp\{iQ\theta\}\ (\partial_\mu + iQ\,\partial_\mu\theta)\ \psi(x)$$

• The 'gauge principle' requires that the $U(1)$ phase invariance should hold *locally*

• To achieve this, one introduces a new spin-1 field $A_\mu(x)$, transforming in such a way as to cancel the $\partial_\mu\theta$ term above

$$A_\mu(x) \quad \xrightarrow{\mathrm{U(1)}} \quad A'_\mu(x) \equiv A_\mu(x) - \frac{1}{e}\,\partial_\mu\theta$$

and defines the covariant derivative

$$D_\mu \psi(x) \equiv [\partial_\mu + ieQA_\mu(x)]\ \psi(x)$$

which transforms like the field itself:

$$D_\mu \psi(x) \quad \xrightarrow{\mathrm{U(1)}} \quad (D_\mu \psi)'(x) \equiv \exp\{iQ\theta\}\, D_\mu \psi(x)$$

• This yields the Lagrangian invariant under local $U(1)$ transformations

$$\mathcal{L} \equiv i\,\overline{\psi}(x)\gamma^\mu D_\mu \psi(x) - m\,\overline{\psi}(x)\psi(x) = \mathcal{L}_0 - eQA_\mu(x)\,\overline{\psi}(x)\gamma^\mu \psi(x)$$

• Gauge invariance has forced an interaction between the Dirac spinor and the gauge field A_μ as in Quantum Electrodynamics (QED)

• For A_μ to be a propagating field, one has to add a gauge-invariant kinetic term

$$\mathcal{L}_{\mathrm{Kin}} \equiv -\frac{1}{4} F_{\mu\nu}(x)\, F^{\mu\nu}(x)$$

where $F_{\mu\nu} \equiv \partial_\nu A_\mu - \partial_\mu A_\nu$ is the usual electromagnetic field strength tensor

• A mass term for the gauge field, $\mathcal{L}_m = \frac{1}{2} m_\gamma^2 A^\mu A_\mu$ forbidden due to gauge invariance

• $\mathcal{L}_{\mathrm{QED}} = i\,\overline{\psi}(x)\gamma^\mu D_\mu \psi(x) - m\,\overline{\psi}(x)\psi(x) - \frac{1}{4} F_{\mu\nu}(x)\, F^{\mu\nu}(x)$

• $\mathcal{L}_{\mathrm{QED}}$ yields Maxwell equations (J^ν is the fermion electromagnetic current)

$$\partial_\mu F^{\mu\nu} = J^\nu \equiv eQ\,\overline{\psi}\gamma^\nu \psi$$

Elementary calculation in QED

Dirac Wave-functions, Matrices and Algebra

- A Dirac spinor for a particle of physical momentum p and polarization s is denoted by $u^s_\alpha(p)$, and for the antiparticle, it is $v^s_\alpha(p)$
  - In each case, the energy $E_0 = +\sqrt{p^2 + m^2}$ is positive
- The general solution of the Dirac equation can be written as a linear combination of plane waves. The positive frequency waves are of the form

$$\psi(x) = u(p)e^{-ip.x}; \quad p^2 = m^2; \quad p^0 > 0$$

  - There are 2 linearly independent solutions of $u(p)$ $s = 1, 2$

$$u^s(p) = \begin{pmatrix} \sqrt{p.\sigma}\xi^s \\ \sqrt{p.\bar{\sigma}}\xi^s \end{pmatrix}$$

  - where $\sigma^\mu =\equiv (1, \sigma^i); \quad \bar{\sigma}^\mu \equiv (1, -\sigma^i)$
- The *negative frequency* solutions of the Dirac equation are

$$\psi(x) = v(p)e^{+ip.x}; \quad p^2 = m^2; \quad p^0 > 0$$

- Dirac equation in terms of the spinor $u^s(p)$ and $v^s(p)$ reads

$$(\not{p} - m)u^s(p) = 0; \quad (\not{p} + m)v^s(p) = 0$$

- In terms of the adjoint spinors $\bar{u} = u^\dagger\gamma^0$ and $\bar{v} = v^\dagger\gamma^0$, the Dirac equation reads

$$\bar{u}^s(p)(\not{p} - m) = 0; \quad \bar{v}^s(p)(\not{p} + m) = 0$$

- They are normalized as: $\bar{u}^r u^s = 2m\delta^{rs}$; $\bar{v}^r v^s = -2m\delta^{rs}$
- They are orthogonal to each other: $\bar{u}^r(p)v^s(p) = \bar{v}(p)u^s(p) = 0$

- Spin Sums:

$$\sum_s = u^s(p)\bar{u}^s(p) = \not{p} + m; \quad \sum_s = v^s(p)\bar{v}^s(p) = \not{p} - m$$

- Completeness relation is:

$$\sum_s [u^s_\alpha(p)\bar{u}^s_\beta(p) - v^s_\alpha(p)\bar{v}^s_\beta(p)] = \delta_{\alpha\beta}$$

- The γ-matrices in the Dirac equation satify the anticommutation relations

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

- A representation useful for the electroweak theory is the Weyl representation:

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix},$$

  where σ are the Pauli matrices and 1 is a 2×2 unit matrix

- Frequently appearing combinations are:

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\mu]; \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5$$

- In this representation

$$\gamma^5 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

- In taking traces in the Dirac space,
  we form hermitian conjugates of matrix elements for which, the following relation holds:

$$[\bar{u}(p', s)\Gamma u(p, s)]^\dagger = \bar{u}(p, s)\bar{\Gamma} u(p', s)$$

  where $\bar{\Gamma} = \gamma^0\Gamma^\dagger\gamma^0$

Feynman Rules

Courtesy: BNL

Differential Cross section

- Expressions for cross sections are divided in two parts:

  1. The invariant amplitude $\mathcal{M}$, which is a Lorentz-scalar and in which physics lies. This is determined by the vertices (interactions) and propagators from space-time point x_1 to x_2, given by the Lagrangian of the theory

  2. The Lorentz-invariant phase space (LIPS), containing the kinematical factors and the correct boundary conditions for a process

- Differential cross section $d\sigma$ for *spinless particles and for photons only*: e.g., $\gamma(p_1) + \gamma(p_2) \longrightarrow P_1(k_1) + ...P_n(k_n)$

$$\begin{aligned} d\sigma &= \frac{1}{|v_1 - v_2|}(\frac{1}{2\omega_{p1}})(\frac{1}{2\omega_{p2}})|\mathcal{M}|^2 \frac{d^3k_1}{2\omega_1(2\pi)^3} ... \frac{d^3k_n}{2\omega_n(2\pi)^3} \\ &\times (2\pi)^4 \delta^4(p_1 + p_2 - \sum_{i=1}^{n} k_i) \end{aligned}$$

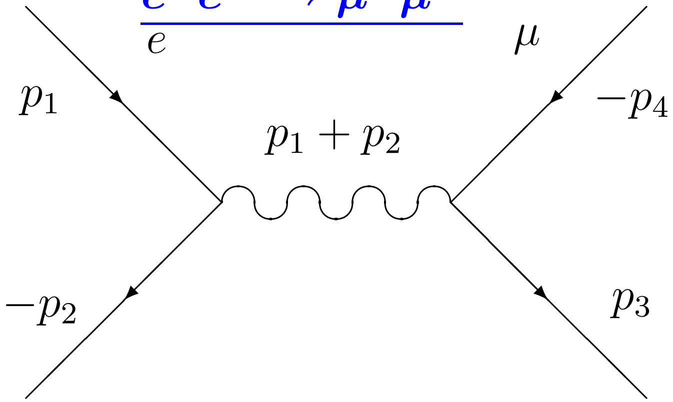
- where $\omega_p = \sqrt{|\vec{p}|^2 + m^2}$, and v_1 and v_2 are the velocities of the incident particles
- The expression is integrated over all undetected momenta $k_1...k_n$
- The statistical fator S is obtained by including a factor $\frac{1}{m!}$: $S = \Pi_i \frac{1}{m_i!}$
- For Dirac particles, the factor $1/2\omega_p$ is replaced by m/E_p
- If desired, polarizations are *summed* over final and averaged over initial states

$e^+e^- \to \mu^+\mu^-$

- Some elementary electromagnetic processes based on the lepton-photon interactions in QED

$$\begin{aligned} \mathcal{L}^{\text{em}} &= +e j^{\text{em}}_{\alpha} A^{\alpha} \\ j^{\text{em}}_{\alpha} &= \bar{\psi}_e \gamma_\alpha \psi_e + \bar{\psi}_\mu \gamma_\alpha \psi_\mu + ... \end{aligned}$$

$\underline{e^+e^- \to \mu^+\mu^-}$



- The above diagram yields the following matrix element (m is the mass of the muon, electron mass is set to zero and $\not{p} = p_\mu \gamma^\mu$)

$$M_{if} = \frac{e^2}{(p_1+p_2)^2} \left[\overline{u}(-p_2)\gamma_\alpha\, u(p_1)\right] \left[\overline{u}(p_3)\gamma^\alpha\, u(-p_4)\right]$$

$$M^*_{if} = \frac{e^2}{(p_1+p_2)^2} \left[\overline{u}(p_1)\gamma_\beta\, u(-p_2)\right] \left[\overline{u}(-p_4)\gamma^\beta\, u(p_3)\right]$$

- Summing up the spins of the leptons leads to the energy projection operators

$$\sum_{s=\pm} \rho^{(s)}(\pm p) = \not{p} \pm m$$

- This yields

$$\frac{1}{4}\sum_{\lambda} |M_{if}|^2 = \frac{e^4}{4(p_1+p_2)^4} Sp\left[\not{p}_2 \gamma^\alpha \not{p}_1 \gamma^\beta\right] Sp\left[(\not{p}_3 + m)\gamma_\alpha(\not{p}_4 - m)\gamma_\beta\right]$$

- Carrying out the traces and doing the algebra

$$(s = 4E^2_{c.m.};\quad t = m^2 - s\cos^2\tfrac{\theta}{2};\quad u = m^2 - s\sin^2\tfrac{\theta}{2})$$

$$\frac{1}{4}\sum_{\lambda} |M_{if}|^2 = \frac{2e^4}{s^2}\left[(m^2-t)^2 + (m^2-u)^2 - 2m^2 s\right]$$

- This leads to the angular distribution ($\alpha = e^2/4\pi$ and $\beta = v/c$)

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{s}\beta\left[1+\cos^2\theta + (1-\beta^2)\sin^2\theta\right]|_{\beta=1} \rightarrow \frac{\alpha^2}{s}\left(1+\cos^2\theta\right)$$

- yielding the cross section

$$\sigma = \frac{4\pi\alpha^2}{3s}$$

- $\frac{d\sigma}{d\Omega}$ and σ have been checked accurately in e^+e^- experiments

Gauge Invariance & Quantum Chromodynamics (QCD)

QCD: A theory of interacting coloured quarks and gluons

Denote by q_f^α a quark field of colour α and flavour f. To simplify the equations, let us adopt a vector notation in colour space: $q_f^T \equiv (q_f^1, q_f^2, q_f^3)$

- The free Lagrangian for the quarks

$$\mathcal{L}_0 = \sum_f \bar{q}_f \left(i\gamma^\mu \partial_\mu - m_f\right) q_f$$

is invariant under arbitrary *global* $SU(3)_C$ transformations in colour space,

$$q_f^\alpha \longrightarrow (q_f^\alpha)' = U^\alpha{}_\beta \, q_f^\beta , \qquad U U^\dagger = U^\dagger U = 1 , \qquad \det U = 1$$

The $SU(3)_C$ matrices can be written in the form

$$U = \exp\left\{ i \frac{\lambda^a}{2} \theta_a \right\}$$

where $\frac{1}{2}\lambda^a$ $(a = 1, 2, \ldots, 8)$ denote the generators of the fundamental representation of the $SU(3)_C$ algebra, and θ_a are arbitrary parameters. The matrices λ^a are traceless and satisfy the commutation relations (f^{abc} the $SU(3)_C$ structure constants, real and antisymmetric)

$$\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2} \right] = i f^{abc} \frac{\lambda^c}{2} ,$$

• Like QED, we require the Lagrangian to be also invariant under *local* $SU(3)_C$ transformations, $\theta_a = \theta_a(x)$. Need to change the quark derivatives by covariant objects.

• Since we have now eight independent gauge parameters, eight different gauge bosons $G^\mu_a(x)$, the so-called *gluons*, are needed:

$$D^\mu q_f \equiv \left[\partial^\mu + i g_s \frac{\lambda^a}{2} G^\mu_a(x)\right] q_f \equiv \left[\partial^\mu + i g_s G^\mu(x)\right] q_f$$

with

$$[G^\mu(x)]_{\alpha\beta} \equiv \left(\frac{\lambda^a}{2}\right)_{\alpha\beta} G^\mu_a(x)$$

• Require that $D^\mu q_f$ transforms in exactly the same way as the colour-vector q_f; this fixes the transformation properties of the gauge fields:

$$D^\mu \longrightarrow (D^\mu)' = U\, D^\mu\, U^\dagger\,, \qquad G^\mu \longrightarrow (G^\mu)' = U\, G^\mu\, U^\dagger + \frac{i}{g_s}(\partial^\mu U)\, U^\dagger$$

• Under an infinitesimal $SU(3)_C$ transformation,

$$q^\alpha_f \longrightarrow (q^\alpha_f)' = q^\alpha_f + i\left(\frac{\lambda^a}{2}\right)_{\alpha\beta} \delta\theta_a\, q^\beta_f\,,$$

$$G^\mu_a \longrightarrow (G^\mu_a)' = G^\mu_a - \frac{1}{g_s}\partial^\mu(\delta\theta_a) - f^{abc}\,\delta\theta_b\, G^\mu_c$$

Observations:

(i) The gauge transformation of the gluon fields is more complicated than the one obtained in QED for the photon. The non-commutativity of the $SU(3)_C$ matrices gives rise to an additional term involving the gluon fields themselves

(ii) For constant $\delta\theta_a$, the transformation rule for the gauge fields is expressed in terms of the structure constants f^{abc}; thus, the gluon fields belong to the adjoint representation of the colour group

(iii) There is a unique $SU(3)_C$ coupling g_s

(iv) To build a gauge-invariant kinetic term for the gluon fields, we introduce the corresponding field strengths:

$$G^{\mu\nu}(x) \equiv -\frac{i}{g_s}[D^\mu, D^\nu] = \partial^\mu G^\nu - \partial^\nu G^\mu + ig_s[G^\mu, G^\nu] \equiv \frac{\lambda^a}{2} G_a^{\mu\nu}(x),$$
$$G_a^{\mu\nu}(x) = \partial^\mu G_a^\nu - \partial^\nu G_a^\mu - g_s f^{abc} G_b^\mu G_c^\nu$$

Under a gauge transformation,

$$G^{\mu\nu} \longrightarrow (G^{\mu\nu})' = U\, G^{\mu\nu}\, U^\dagger$$

and the colour trace Tr $(G^{\mu\nu}G_{\mu\nu}) = \frac{1}{2} G_a^{\mu\nu} G^a_{\mu\nu}$ remains invariant

SU(N) Algebra

- $SU(N)$ is the group of $N \times N$ unitary matrices, $UU^\dagger = U^\dagger U = 1$, with $\det U = 1$
- The generators of the $SU(N)$ algebra, T^a $(a = 1, 2, \ldots, N^2 - 1)$, are hermitian, traceless matrices satisfying the commutation relations

$$[T^a, T^b] = i f^{abc} T^c$$

f^{abc} are structure constants; they are real and totally antisymmetric

- The fundamental representation $T^a = \lambda^a/2$ is N–dimensional. For $N = 2$, λ^a are the usual Pauli matrices, while for $N = 3$, they are the eight Gell-Mann matrices

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

They satisfy the anticommutation relation

$$\{\lambda^a, \lambda^b\} = \frac{4}{N} \delta^{ab} I_N + 2 d^{abc} \lambda^c$$

I_N is the N–dimensional unit matrix; d^{abc} are totally symmetric in the three indices.

• For $SU(3)$, the only non-zero (up to permutations) f^{abc} and d^{abc} constants are

$$\frac{1}{2}f^{123} = f^{147} = -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367}$$
$$= \frac{1}{\sqrt{3}}f^{458} = \frac{1}{\sqrt{3}}f^{678} = \frac{1}{2}$$
$$d^{146} = d^{157} = -d^{247} = d^{256} = d^{344} = d^{355} = -d^{366} = -d^{377} = \frac{1}{2}$$
$$d^{118} = d^{228} = d^{338} = -2d^{448} = -2d^{558} = -2d^{688} = -2d^{788} = -d^{888} = \frac{1}{\sqrt{3}}$$

• The adjoint representation of the $SU(N)$ group is given by the $(N^2-1)\times(N^2-1)$ matrices $(T_A^a)_{bc} \equiv -i f^{abc}$

• Relations defining the $SU(N)$ invariants T_F, C_F and C_A

$$\mathrm{Tr}\left(\lambda^a\lambda^b\right) = 4\,T_F\,\delta_{ab}\,, \qquad T_F = \frac{1}{2}$$
$$\left(\lambda^a\lambda^a\right)_{\alpha\beta} = 4C_F\,\delta_{\alpha\beta}\,, \qquad C_F = \frac{N^2-1}{2N}$$
$$\mathrm{Tr}(T_A^aT_A^b) = f^{acd}f^{bcd} = C_A\,\delta_{ab} \qquad C_A = N$$

Gauge fixing and Ghost Fields

• The fields G_a^μ have 4 Lorentz degrees of freedom, while a massless spin-1 gluon has 2 physical polarizations only. The unphysical degrees of freedom have to be removed

• The canonical momentum associated with G_a^μ, $\Pi_\mu^a(x) \equiv \delta\mathcal{L}_{\rm QCD}/\delta(\partial_0 G_a^\mu) = G_{\mu 0}^a$, vanishes identically for $\mu = 0$. The commutation relation meaningless for $\mu = \nu = 0$

$$\left[G_a^\mu(x), \Pi_b^\nu(y)\right]\delta(x^0 - y^0) \;=\; i g^{\mu\nu}\delta^{(4)}(x-y)\delta_{ab}$$

• Hence, the unphysical components of the gluon field should not be quantized. This could be achieved by imposing two gauge conditions, such as $G_a^0 = 0$ and $\vec{\nabla}.\vec{G}_a = 0$. This is a (Lorentz) non-covariant procedure, which leads to an awkward formalism

• Instead, one can impose a Lorentz–invariant gauge condition, such as $\partial_\mu G_a^\mu = 0$ by adding to the Lagrangian the gauge-fixing term (ξ is the gauge parameter)

$$\mathcal{L}_{\rm GF} \;=\; -\frac{1}{2\xi}\left(\partial^\mu G_\mu^a\right)\left(\partial_\nu G_a^\nu\right)$$

• The 4 Lorentz components of the canonical momentum are then non-zero and one can develop a covariant quantization formalism

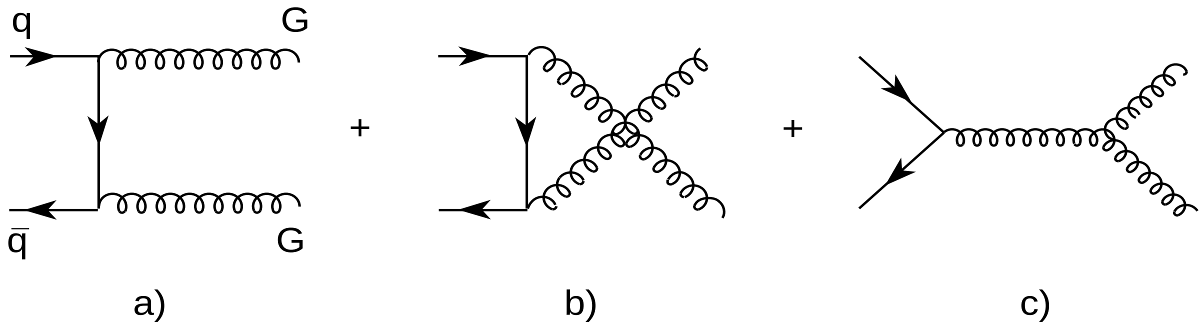
$$\Pi_\mu^a(x) \equiv \frac{\delta\mathcal{L}_{\rm QCD}}{\delta(\partial_0 G_a^\mu)} = G_{\mu 0}^a - \frac{1}{\xi}\, g_{\mu 0}\left(\partial^\nu G_\nu^a\right)$$

• Since $\mathcal{L}_{\mathrm{GF}}$ is a quadratic G_a^μ term, it modifies the gluon propagator:

$$\langle 0|T[G_a^\mu(x)G_b^\nu(y)]|0\rangle = i\delta_{ab}\int \frac{d^4k}{(2\pi)^4}\frac{\mathrm{e}^{-ik(x-y)}}{k^2+i\varepsilon}\left\{-g^{\mu\nu}+(1-\xi)\frac{k^\mu k^\nu}{k^2+i\varepsilon}\right\}.$$

• In QED, this gauge-fixing procedure is enough for making a consistent quantization of the theory. In non-abelian gauge theories, like QCD, a second problem still remains

• Let us consider the scattering process $q\bar{q}\to GG$, for which the scattering amplitude has the general form $T = J^{\mu\mu'}\varepsilon_\mu^{(\lambda)}\varepsilon_{\mu'}^{(\lambda')}$



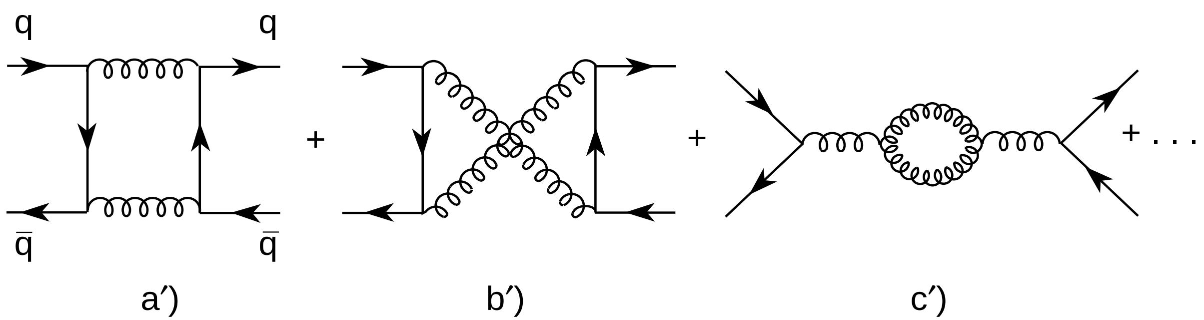
• The probability associated with the scattering process is:

$$\mathcal{P}\sim\frac{1}{2}J^{\mu\mu'}(J^{\nu\nu'})^\dagger\sum_\lambda \varepsilon_\mu^{(\lambda)}\varepsilon_\nu^{(\lambda)*}\sum_{\lambda'}\varepsilon_{\mu'}^{(\lambda')}\varepsilon_{\nu'}^{(\lambda')*}$$

This involves a sum over the final gluon polarizations

• The physical probability $\mathcal{P}_T$, where only the two transverse gluon polarizations are considered in the sum, is different from the covariant quantity $\mathcal{P}_C$, which includes a sum over all polarization components: $\mathcal{P}_C > \mathcal{P}_T$. As only $\mathcal{P}_T$ has physical meaning, one should just sum over the physical transverse polarizations to get the right answer

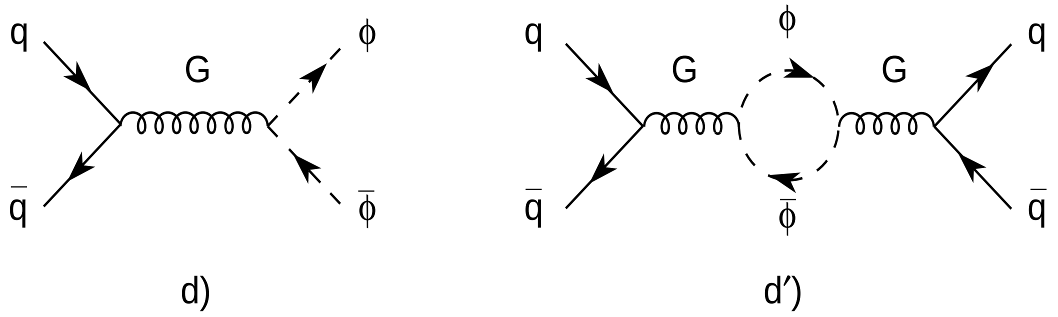
• However, higher-order graphs such as the ones shown below get unphysical contributions from the longitudinal and scalar gluon polarizations propagating along the internal gluon lines, implying a violation of unitarity (the two fake polarizations contribute a positive probability)



• In QED this problem does not appear because the gauge-fixing condition $\partial^\mu A_\mu = 0$ respects the conservation of the electromagnetic current due to the extra condition $\Box\theta = 0$, i.e., $\partial_\mu J^\mu_{\rm em} = \partial_\mu(eQ\bar{\Psi}\gamma^\mu\Psi) = 0$, and therefore, $\mathcal{P}_C = \mathcal{P}_T$.

• In QCD, $\mathcal{P}_C \neq \mathcal{P}_T$ stems from the third diagram in the figure above involving a gluon self-interaction. The gauge-fixing condition $\partial_\mu G^\mu_a = 0$ does not leave any residual invariance. Thus, $k_\mu J^{\mu\mu'} \neq 0$

• A clever solution (due to Feynman; 1963) is to add unphysical fields, the so-called *ghosts*, with a coupling to the gluons so as to exactly cancel *all* the unphysical contributions from the scalar and longitudinal gluon polarizations



• Since a positive fake probability has to be cancelled, one needs fields obeying the wrong statistics (i.e., of negative norm) and thus giving negative probabilities. The cancellation is achieved by adding to the Lagrangian the Faddeev–Popov term (Fadeev, Popov; 1967),

$$\mathcal{L}_{\mathrm{FP}} = -\partial_\mu \bar{\phi}_a D^\mu \phi^a \,, \qquad D^\mu \phi^a \equiv \partial^\mu \phi^a - g_s f^{abc} \phi^b G_c^\mu$$

where $\bar{\phi}^a$, ϕ^a $(a = 1, \ldots, N_C^2 - 1)$ is a set of anticommuting (obeying the Fermi–Dirac statistics), massless, hermitian, scalar fields. The covariant derivative $D^\mu \phi^a$ contains the needed coupling to the gluon field. One can easily check that finally $\mathcal{P}_C = \mathcal{P}_T$.

• Thus, the addition of the gauge-fixing and Faddeev–Popov Lagrangians allows to develop a simple covariant formalism, and therefore a set of simple Feynman rules

QCD Lagrangian in a Covariant Gauge

- Putting all pieces together

$$\mathcal{L}_{\rm QCD} = \mathcal{L}_{\rm (q,g)} + \mathcal{L}_{\rm GF} + \mathcal{L}_{\rm FP}$$

$$\begin{aligned}\mathcal{L}_{\rm (q,g)} \;&=\; -\frac{1}{4}\left(\partial^\mu G_a^\nu - \partial^\nu G_a^\mu\right)\left(\partial_\mu G_\nu^a - \partial_\nu G_\mu^a\right) + \sum_f \bar{q}_f^\alpha \left(i\gamma^\mu \partial_\mu - m_f\right) q_f^\alpha \\ &- g_s\, G_a^\mu \sum_f \bar{q}_f^\alpha\, \gamma_\mu \left(\frac{\lambda^a}{2}\right)_{\alpha\beta} q_f^\beta \\ &+ \frac{g_s}{2} f^{abc}\left(\partial^\mu G_a^\nu - \partial^\nu G_a^\mu\right) G_\mu^b\, G_\nu^c - \frac{g_s^2}{4} f^{abc} f_{ade}\, G_b^\mu\, G_c^\nu\, G_\mu^d\, G_\nu^e\end{aligned}$$

$$\mathcal{L}_{\rm GF} \;=\; -\frac{1}{2\xi}\left(\partial^\mu G_\mu^a\right)\left(\partial_\nu G_a^\nu\right)$$

$$\mathcal{L}_{\rm FP} \;=\; -\partial_\mu \bar{\phi}_a D^\mu \phi^a\,, \qquad D^\mu \phi^a \equiv \partial^\mu \phi^a - g_s f^{abc} \phi^b G_c^\mu$$

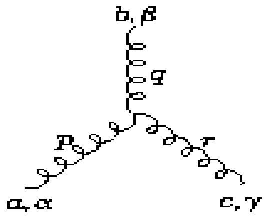
Feynman Rules for QCD

Propagators and Vertices

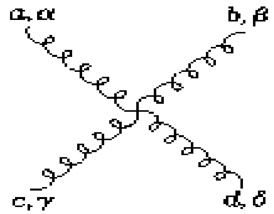
$$= \delta^{ab}\frac{-i\,g^{\alpha\beta}}{p^2+i\epsilon} \quad \text{(Feynman gauge)}$$

$$= \delta^{ab}\frac{i}{p^2+i\epsilon}$$

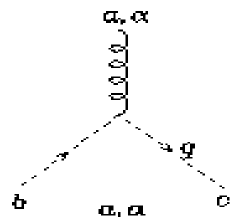
$$= \delta^{ik}\left.\frac{i}{\not p - m + i\epsilon}\right|_{mn}$$



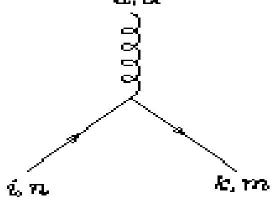
$$= g\,f^{abc}\left[g^{\alpha\beta}(p-q)^{\gamma} + g^{\beta\gamma}(q-r)^{\alpha} + g^{\gamma\alpha}(r-p)^{\beta}\right]$$



$$\begin{aligned} = &-ig^2 f^{xac} f^{xbd}\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\beta\gamma}\right) \\ &-ig^2 f^{xad} f^{xbc}\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta}\right) \\ &-ig^2 f^{xab} f^{xcd}\left(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}\right) \end{aligned}$$



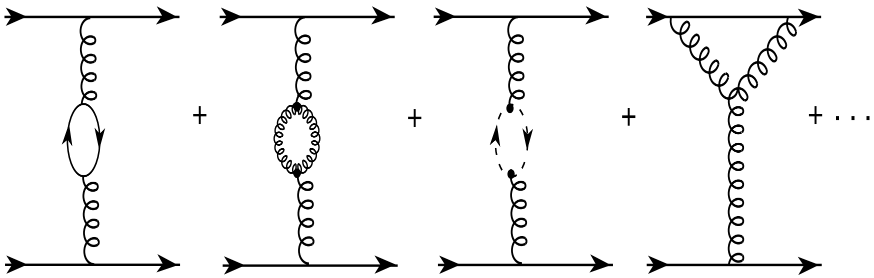
$$= -g\,f^{abc}\,q^{\alpha}$$



$$= i g\,\lambda^{a}_{ki}\,\gamma^{\alpha}_{mn}$$

The QCD running coupling constant

- The renormalization of the QCD coupling proceeds in a way similar to QED. Owing to the non-abelian character of $SU(3)_C$, there are additional contributions involving gluon self-interactions (and ghosts)



- The scale dependence of $\alpha_s(Q^2)$ is regulated by the so-called β function of a theory

$$\mu \frac{d\alpha_s}{d\mu} \equiv \alpha_s \beta(\alpha_s); \quad \beta(\alpha_s) = \beta_1 \frac{\alpha_s}{\pi} + \beta_2 (\frac{\alpha_s}{\pi})^2 + ...$$

- From the above one-loop diagrams, one gets the value of the first β function coefficient [Politzer; Gross & Wilczek; 1973]

$$\beta_1 = \frac{2}{3} T_F N_f - \frac{11}{6} C_A = \frac{2\,N_f - 11\,N_C}{6}$$

- The positive contribution proportional to the number of quark flavours N_f is generated by the q–$\bar{q}$ loops; the gluonic self-interactions introduce the additional negative contribution proportional to N_C

- The term proportional to N_C is responsible for the completely different behaviour of QCD ($N_C = 3$):

$$\beta_1 < 0 \text{ if } N_f \leq 16$$

The corresponding QCD running coupling, $\alpha_s(Q^2)$, decreases at short distances:

$$\lim_{Q^2 \to \infty} \alpha_s(Q^2) \to 0$$

For $N_f \leq 16$, QCD has the property of asymptotic freedom

- Quantum effects have introduced a dependence of the coupling on the energy; still need a reference scale to decide when a given Q^2 can be considered large or small

- A precise definition of the scale is obtained from the solution of the β-function differential equation. At one loop, one gets

$$\ln \mu + \frac{\pi}{\beta_1 \alpha_s(\mu^2)} = \ln \Lambda_{\rm QCD}$$

where $\ln \Lambda_{\rm QCD}$ is an integration constant. Thus,

$$\alpha_s(\mu^2) = \frac{2\pi}{-\beta_1 \ln\left(\mu^2/\Lambda^2_{\rm QCD}\right)}$$

- Dimensional Transmutation: The dimensionless parameter g_s is traded against the dimensionful scale $\Lambda_{\rm QCD}$ generated by Quantum effects

- For $\mu \gg \Lambda_{\rm QCD}$, $\alpha_s(\mu^2) \longrightarrow 0$, one recovers asymptotic freedom
- At lower energies, the running coupling gets larger; for $\mu \longrightarrow \Lambda_{\rm QCD}$, $\alpha_s(\mu^2) \longrightarrow \infty$, perturbation theory breaks down. The QCD regime $\alpha_s(\mu^2) \geq 1$ requires non-perturbative methods, such as Lattice-QCD (beyond the scope of these lectures)

Higher Orders

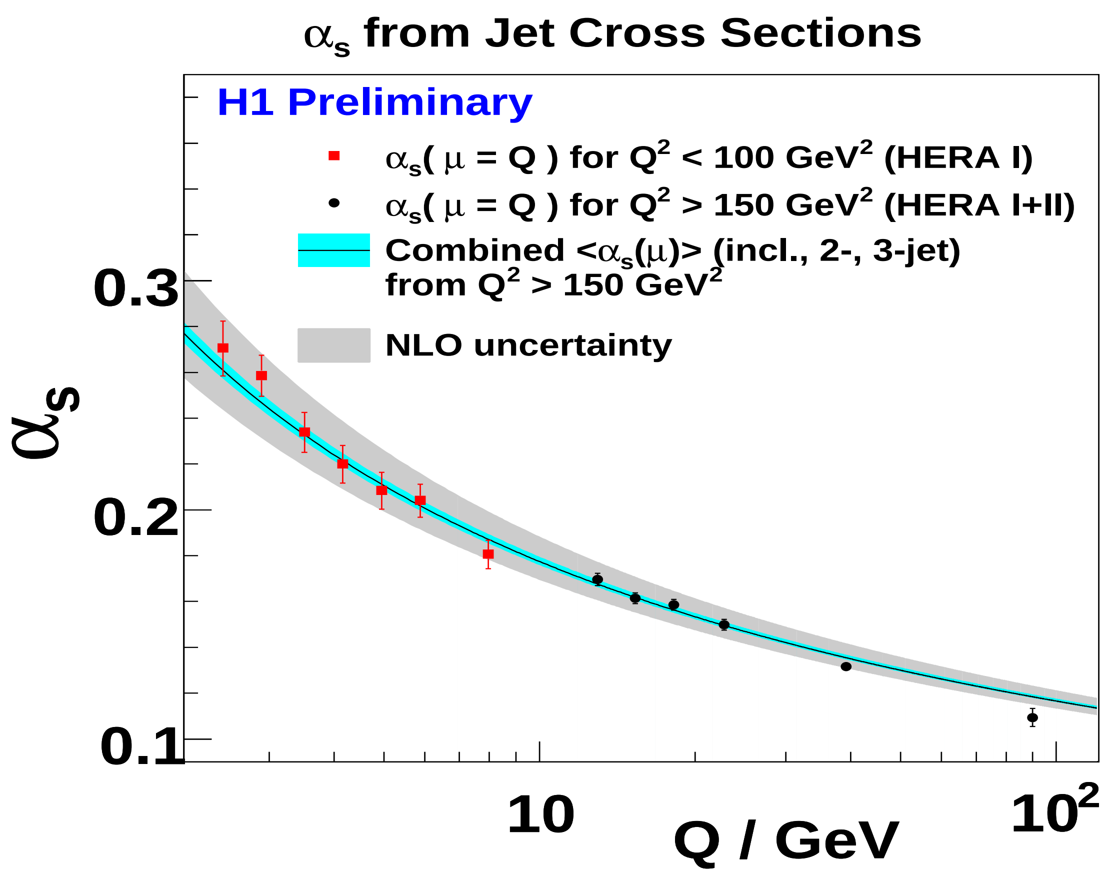
- Higher orders in perturbation theory are much more important in QCD than in QED, because the coupling is much larger (at ordinary energies)
- In the meanwhile, the β function is known to four loops $\beta(\alpha_s) = \sum_{i=1}^{N} \beta_k (\frac{\alpha_s}{\pi})^k$ [van Ritbergen, Vermaseren, Larin [1997]; Caswell; D.R.T. Jones [1974]; Vladimirov, Zakharov [1980]]
- In the $\overline{\rm MS}$ scheme, the computed higher-order coefficients take the values ($\zeta_3 = 1.202056903\ldots$)

$$\beta_2 = -\frac{51}{4} + \frac{19}{12} N_f\,; \qquad \beta_3 = \frac{1}{64}\left[-2857 + \frac{5033}{9} N_f - \frac{325}{27} N_f^2\right]\,;$$

$$\beta_4 = \frac{-1}{128}\left[\left(\frac{149753}{6} + 3564\,\zeta_3\right) - \left(\frac{1078361}{162} + \frac{6508}{27}\,\zeta_3\right) N_f + \left(\frac{50065}{162} + \frac{6472}{81}\,\zeta_3\right) N_f^2 + \frac{1093}{729} N_f^3\right]$$

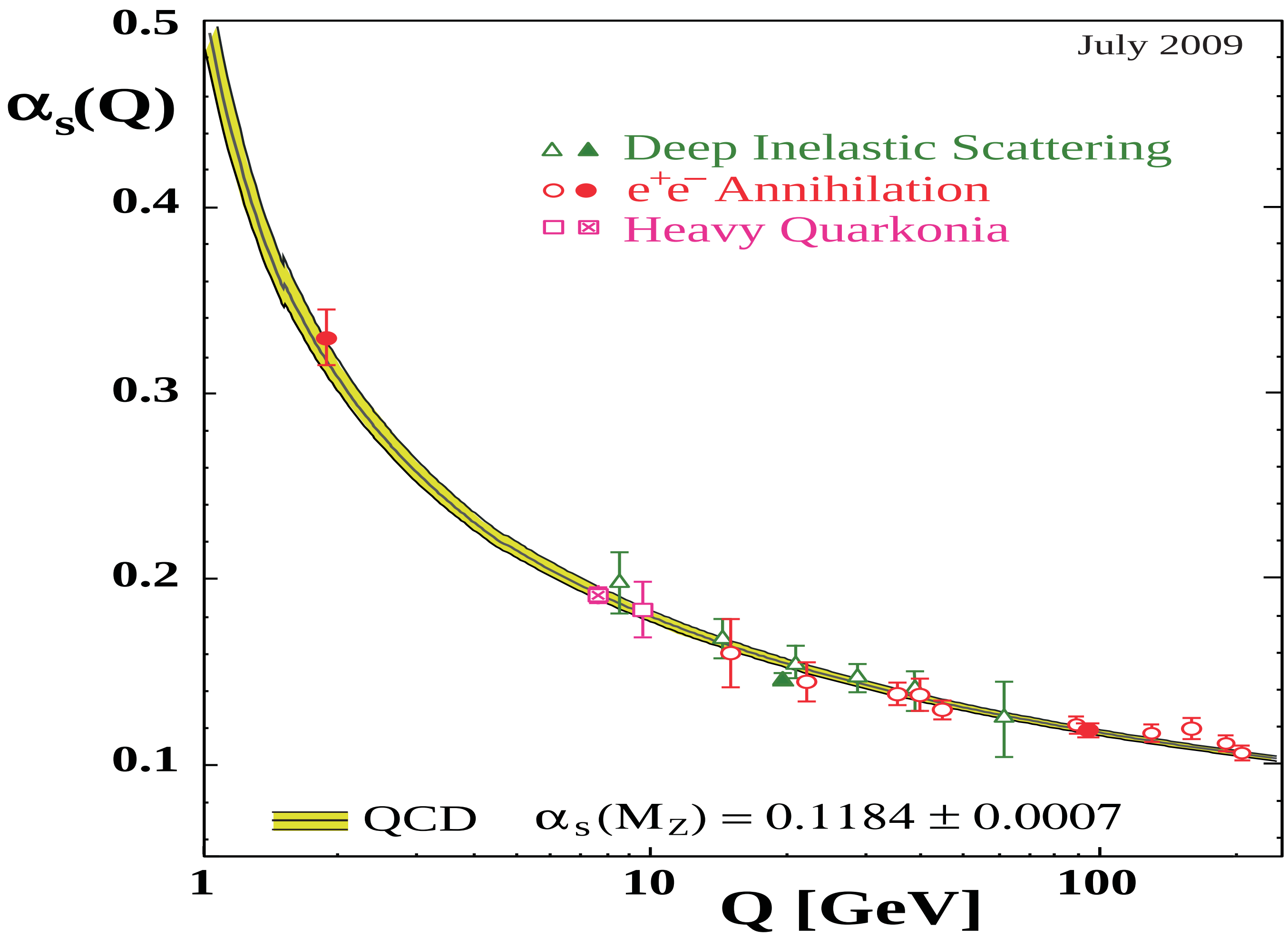
Measurements of $\alpha_s(Q^2)$ from jets at HERA/H1

H1 Collaboration (DESY 08-032)



Experimental Evidence of Asymptotic Freedom

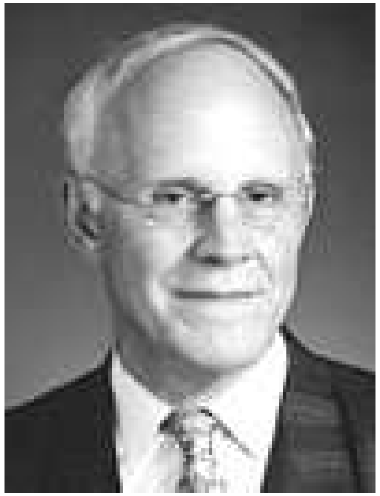
S. Bethke, Summer 2009



Nobel Prize for Physics 2004

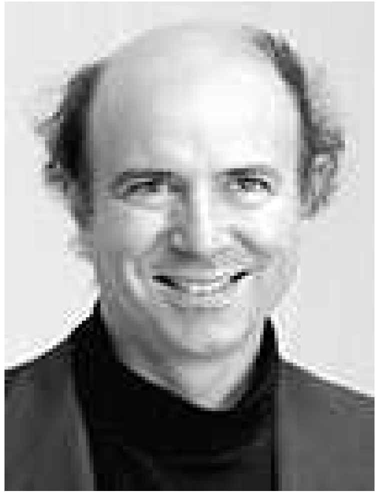
Nobelprize.org

"for the discovery of asymptotic freedom in the theory of the strong interaction"



David J. Gross
1/3 of the prize
USA

H. David Politzer
1/3 of the prize
USA

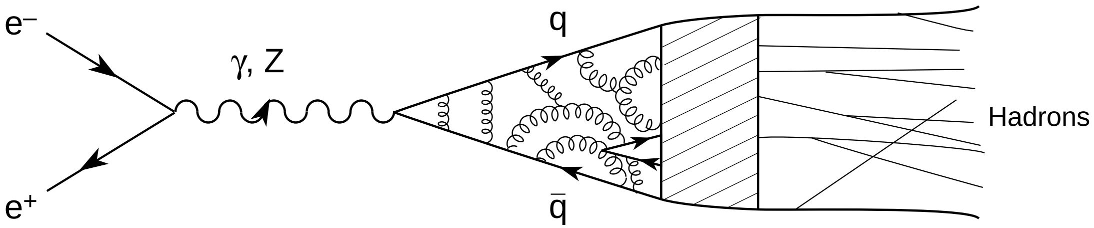


Frank Wilczek
1/3 of the prize
USA

Applications of perturbative QCD to hard processes

- The simplest hard process is the total hadronic cross-section in e^+e^- annihilation:

$$R \equiv \sigma(e^+e^- \to \mathbf{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$$

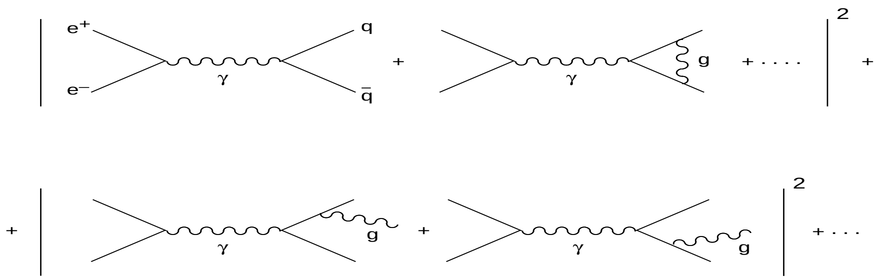


- In the quark-parton model (i.e., no QCD corrections), one has for $(i = u, d, c, s, b)$ and taking into account only the γ-exchange in the intermediate state ($N_c = 3$):

$$R = N_c \sum_i Q_i^2 = N_c \frac{11}{9} = \frac{11}{3}$$

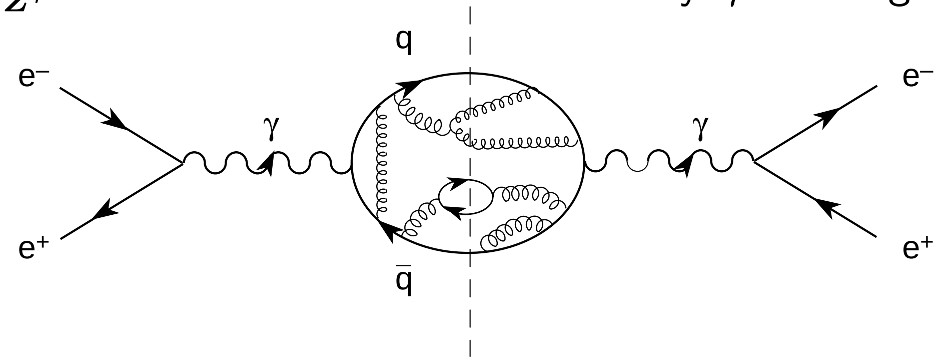
- In leading order perturbative QCD, the process is approximated by the following

$$\sigma(e^+e^- \to \mathbf{hadrons}) = \sigma(e^+e^- \to q\bar{q} + q\bar{q}g)$$



The Photon vacuum polarization amplitude $\Pi_{\rm em}(q^2)$

- For $E^2_{\rm c.m.} \ll M_Z^2$, $e^+e^- \to$ hadrons dominated by γ-exchange



- Easier to calculate the so-called Photon vacuum polarization amplitude, $\Pi_{\rm em}(q^2)$

- $\Pi_{\rm em}(q^2)$ is defined by the time-ordered product of two electromagnetic currents $J_\mu^{\rm em} = \sum_f Q_f \bar{q}_f \gamma_\mu q_f$: [a]

$$\Pi_{\mu\nu}(q) = i\int d^4x e^{iq.x}\langle 0|TJ_\mu^{\rm em}(x)J_\nu^{\rm em}(0)|0\rangle = (-g_{\mu\nu}q^2 + q_\mu q_\nu)\Pi_{\rm em}(q^2)$$

$$R_{e^+e^-}(s) \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 12\pi\ {\rm Im}\Pi_{\rm em}(s)$$

[a] Time-ordered product is defined as: $T\phi(x')\phi(x) = \theta(t'-t)\phi(x')\phi(x) + \theta(t-t')\phi(x)\phi(x')$

$R_{e^+e^-}(s)$ in pert. QCD

- One can compute, in principle, the cross-section of all subprocesses at a given order in α_s

$$e^+e^- \to \gamma^* \to q\bar{q} + q\bar{q}g + q\bar{q}gg + q\bar{q}q'\bar{q}' + \ldots$$

$$R_{e^+e^-}(s) = \left(\sum_{f=1}^{N_f} Q_f^2\right) N_C \left\{1 + \sum_{n\geq 1} F_n \left(\frac{\alpha_s(s)}{\pi}\right)^n\right\}$$

- So far, perturbative series has been calculated to $\mathcal{O}(\alpha_s^3)$:

$$F_1 = 1\,, \qquad F_2 = 1.986 - 0.115\, N_f\,,$$

$$F_3 = -6.637 - 1.200\, N_f - 0.005\, N_f^2 - 1.240\, \frac{\left(\sum_f Q_f\right)^2}{3\sum_f Q_f^2}$$

- For $N_f = 5$ flavours, one has:

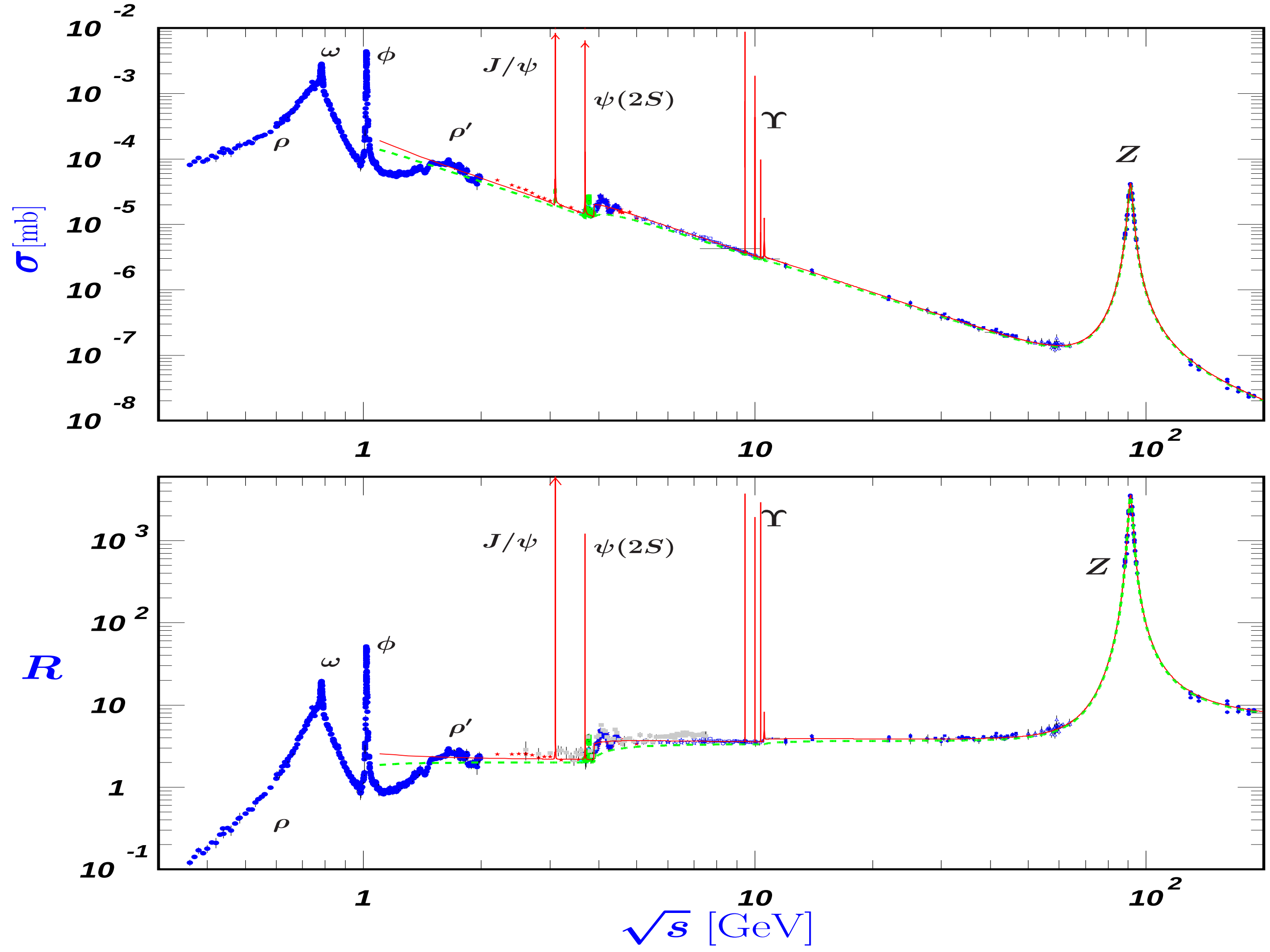
$$R_{e^+e^-}(s) = \frac{11}{3}\left\{1 + \frac{\alpha_s(s)}{\pi} + 1.411\left(\frac{\alpha_s(s)}{\pi}\right)^2 - 12.80\left(\frac{\alpha_s(s)}{\pi}\right)^3 + \mathcal{O}(\alpha_s^4)\right\}$$

- A fit to e^+e^- data for $\sqrt{s}$ between 20 and 65 GeV yields [D. Haidt, 1995]

$$\alpha_s(35\ \text{GeV}) = 0.146 \pm 0.030$$

$R_{e^+e^-}(s)$: Data

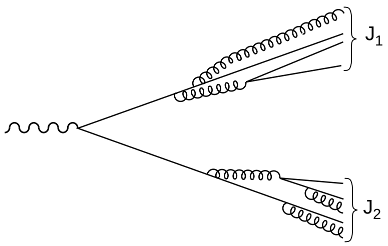
[PDG 2007; W.-M. Yao et al., Journal of Physics, G 33, 1 (2006)]



$e^+e^- \to$ jets

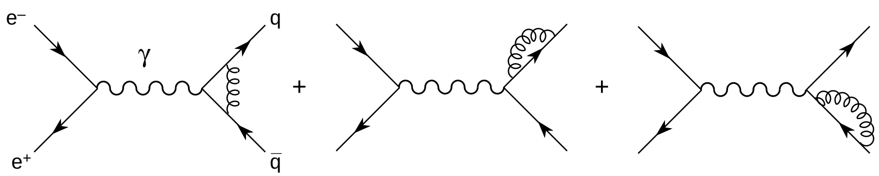
- Quark jets were discovered in 1975 in e^+e^- annihilation experiment at SLAC [R.F. Schwitters et al., Phys. Rev. Lett. **35** (1975) 1320; G.G. Hanson et al., Phys. Rev. Lett. **35** (1975) 1609]

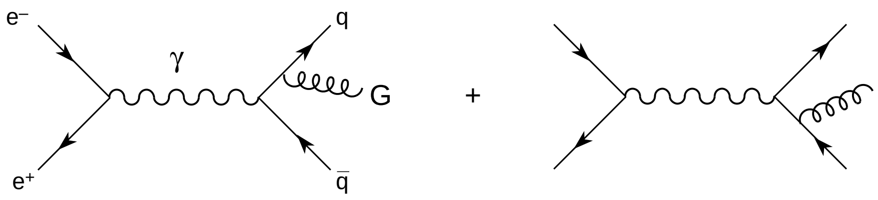
- In QCD, this process is described by $e^+e^- \to q\bar{q}$, radiation of soft partons (gluons, $q\bar{q}$ pairs) and subsequent hadronization



- One needs a definition of hadronic jets, e.g., a cone with a minimum fractional energy and angular resolution (ϵ, δ), or a minimum invariant mass $(y_{\min} = m^2_{\min}(\text{jet})/s)$ to satisfy the Bloch-Nordsiek and KLN theorems

- In $\mathcal{O}(\alpha_s)$ pert. QCD, following Feynman diagrams are to be calculated:





- The process $e^+e^- \longrightarrow \mathbf{2}$ **jets** was first calculated in $\mathcal{O}(\alpha_s)$ by Sterman and Weinberg using the (ϵ, δ)-prescription of jets
[G. Sterman and S. Weinberg, Phys. Rev. Lett. **39** (1977) 1436]

- To $\mathcal{O}(\epsilon), \mathcal{O}(\delta)$:

$$\sigma_{2-\text{jets}}(\epsilon, \delta) = \sigma_0[1 + C_F \frac{\alpha_s(Q^2)}{\pi} \left(-4 \ln \epsilon \ln \delta^2 - 3 \ln \delta^2 - \frac{2\pi^2}{3} + 5 \right)]$$

where $\sigma_0 = \frac{4\pi\alpha^2}{3s} N_C \sum_i Q_i^2; \quad s = Q^2$

- Gluon jets were discovered in 1979 in e^+e^- annihilation experiment at DESY
[R. Brandelik et al. (TASSO Coll.), Phys. Lett. **86B** (1979) 243; D.P. Barber et al. (MARK J Coll.), Phys. Rev. Lett. **43** (1979) 830; and subsequently by Ch. Berger et al. (PLUTO Coll.), Phys. Lett. **86B** (1979) 418 and W. Bartel et al. (JADE Coll.), Phys. Lett. **91B** (1980) 142]

- In $\mathcal{O}(\alpha_s)$, the emission of a hard (i.e., energetic and non-collinear) gluon from a quark leg leads to a three-jet event.

- For massless quarks, the differential distribution is [J. Ellis, M.K. Gaillard, G.G. Ross, Nucl. Phys. **B111** (1976) 253]

$$\frac{1}{\sigma_0}\frac{d^2\sigma}{dx_1 dx_2} = \frac{2\alpha_s}{3\pi}\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$$

and the kinematics is defined as: $x_i = 2\frac{E_i}{\sqrt{s}}$ $(i=1,2,3)$ and $x_1+x_2+x_3=2$

- Defining a jet by an invariant–mass cut y:

$$\mathbf{3\ jet} \qquad \Longleftrightarrow \qquad s_{ij} \equiv (p_i+p_j)^2 > ys \qquad (\forall i,j=1,2,3)$$

the fraction of 3-jet events is predicted to be: $(\mathrm{Li}_2(z) \equiv -\int_0^z \frac{d\xi}{1-\xi}\ln\xi)$ [See, G. Kramer, Springer Tracts in Modern Physics, Vol. 102]

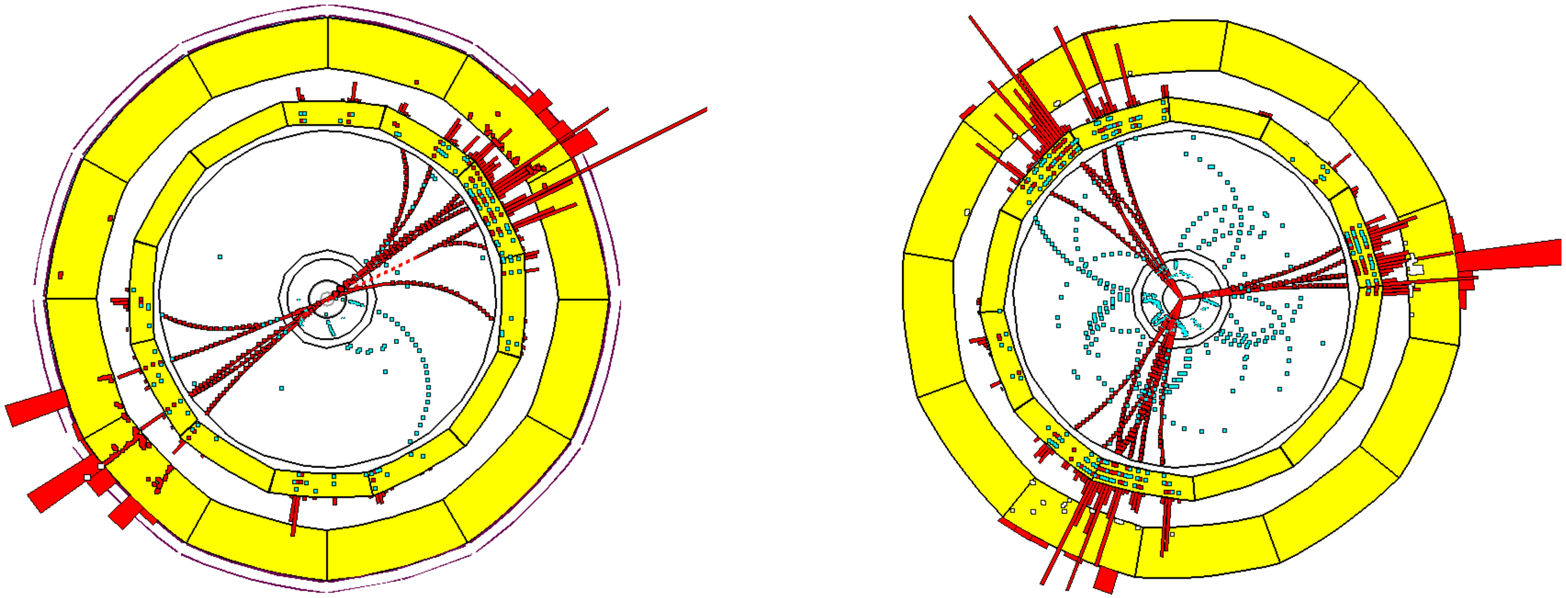
$$R_3(s,y) = \frac{2\alpha_s}{3\pi}\left\{(3-6y)\ln\left(\frac{y}{1-2y}\right) + 2\ln^2\left(\frac{y}{1-y}\right) + \frac{5}{2} - 6y - \frac{9}{2}y^2 + 4\mathrm{Li}_2\left(\frac{y}{1-y}\right) - \frac{\pi^2}{3}\right\}$$

- The corresponding fraction of 2-jet events is given by $R_2 = 1 - R_3$. The general expression for the fraction of n-jet events takes the form (with $\sum_n R_n = 1$):

$$R_n(s,y) = \left(\frac{\alpha_s(s)}{\pi}\right)^{n-2}\sum_{j=0} C_j^{(n)}(y)\left(\frac{\alpha_s(s)}{\pi}\right)^j$$

Two- and Three-jet events from $Z \to q\bar{q}$ and $Z \to q\bar{q}g$

[ALEPH Coll., http://aleph.web.cern.ch/aleph/dali/ Z0_examples.html]



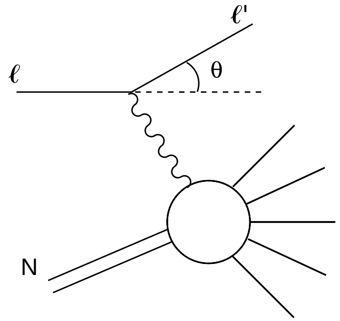
- The jet fractions have a high sensitivity to α_s [$R_n \sim \alpha_s^{n-2}$]. Measurements at the Z peak, from LEP and SLC, using resummed $\mathcal{O}(\alpha_s^2)$ fits to a large set of shape variables, yields [PDG 2006]:

$$\alpha_s(M_Z) = 0.1202 \pm 0.005$$

Deep Inelastic Scattering and QCD

- An important class of hard processes is Deep Inelastic Scattering (DIS) off a Nucleon

$$\ell(k) + N(p) \to \ell'(k') + X(p_X); \quad \ell = e^\pm, \mu^\pm, \nu, \bar{\nu}$$



- Kinematics: p is the momentum of the nucleon having a mass M and the momentum of the hadronic system is p_X; the virtual momentum q of the gauge boson is spacelike

$$Q^2 = -q^2 = -(k-k')^2 = 4EE' \sin^2(\theta/2); \quad s = (p+k)^2; W^2 = p_X^2; \; \nu = (p.q)/M$$

- Useful to define scaling variables; Bjorken-variable x and y

$$x = \frac{-(k-k')^2}{2p.(k-k')} = \frac{Q^2}{2M\nu} = \frac{Q^2}{W^2 + Q^2 - M^2}$$

$$y = \frac{p.(k-k')}{p.k} = \frac{2M\nu}{(s-M^2)} = \frac{W^2 + Q^2 - M^2}{(s-M^2)}$$

- Bjorken limit is defined by: large Q^2 and ν such that x is finite

Cross-sections and Structure functions

- The cross-section for the process $\ell(k) + N(p) \to \ell'(k') + X(p_X)$ mediated by a virtual photon is (M is the nucleon mass; $Q^2 = -q^2$ and q^2 is the photon virtuality)

$$k'_0 \frac{d\sigma}{d^3k'} = \frac{2M}{(s-M^2)} \frac{\alpha^2}{Q^4} L^{\mu\nu} W_{\mu\nu}$$

- $L^{\mu\nu}$ is obtained from the leptonic electromagnetic vertex

$$L^{\mu\nu} = \frac{1}{2} Tr(\not{k}'\gamma_\mu \not{k}\gamma_\nu) = 2(k_\mu k'_\nu + k'_\mu k_\nu - \frac{1}{2} Q^2 g_{\mu\nu})$$

- The strong interaction aspects (involving the structure of the nucleon) is contained in the tensor $W_{\mu\nu}$, which is defined as

$$W_{\mu\nu} = \int dx \mathrm{e}^{iqx} \ < p|J_\mu^\dagger(x) J_\nu(0)|p >$$

- Structure functions (SF) are defined from the general form of $W_{\mu\nu}$ using Lorentz invariance and current conservation. For the electromagnetic currents between unpolarized nucleons,one has

$$W_{\mu\nu} = -(g_{\mu\nu} + \frac{q_\mu q_\nu}{Q^2}) W_1(\nu, Q^2) + (p_\mu + \frac{p.q}{Q^2} q_\mu)(p_\nu + \frac{p.q}{Q^2} q_\nu) \frac{W_2(\nu, Q^2)}{M^2}$$

- This leads to the differential cross-section in terms of x and y:

$$\frac{d^2\sigma}{dxdy} = \frac{2\pi\alpha^2 M}{(s-M^2)x}\left[2W_1(x,y) + W_2(x,y)\left\{\frac{(s-M^2)}{M^2}\frac{(1-y)}{xy} - 1\right\}\right]$$

- The SFs W_1 and W_2 are related to the absorption cross-sections for virtual transverse (σ_T) and longitudinal photons (σ_0)

$$\begin{aligned} W_1 &= \frac{\sqrt{\nu^2+Q^2}}{4\pi\alpha^2}\sigma_T \\ W_2 &= \frac{Q^2}{4\pi\alpha^2\sqrt{\nu^2+Q^2}}(\sigma_0+\sigma_T) \end{aligned}$$

- One can define a longitudinal structure function W_L by

$$W_L = \frac{\sqrt{\nu^2+Q^2}}{4\pi\alpha^2}\sigma_0 = (1+\frac{\nu^2}{Q^2})W_2 - W_1$$

- In the Quark-Parton Model, SFs become scale-invariant in the Bjorken limit

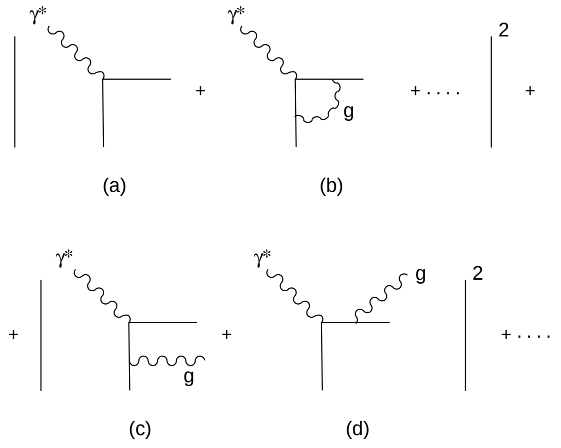
$$\begin{aligned} MW_1(\nu,Q^2) &\rightarrow F_1(x) = \sum_i e_i^2 q_i(x) \\ \nu W_2 &= 2xMW_1 \Longrightarrow \nu W_2(\nu,Q^2) \rightarrow F_2(x) = \sum_i e_i^2 x q_i(x) \end{aligned}$$

- $q_i(x)$ is the probability of finding charged partons (i.e., quarks) inside the nucleon having a fractional momentum x

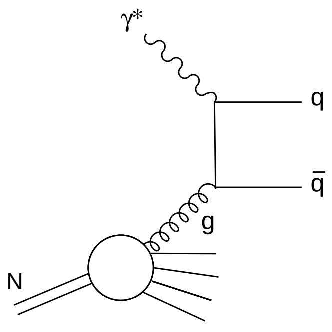
QCD-improved parton model and DIS

- In QCD, one has also gluons as partons, and one has efficient radiation of hard gluons from the struck quarks and gluons in the nucleons

$\mathcal{O}(\alpha_s)$ corrections to the Quark-Parton Model



DIS off a gluon in the nucleon



QCD-improved parton model and DIS -Contd.

- Radiation of gluons produces the evolution of the structure functions. As Q^2 increases, more and more gluons are radiated, which in turn split into $q\bar{q}$ pairs
- This process leads to the softening of the initial quark momentum distributions and to the growth of the gluon density and the $q\bar{q}$ sea for low values of x. Both effects have been firmly established at HERA
- In pert. QCD, these effects can be calculated in terms of the evolution governed by the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi or DGLAP equations
  DGLAP Equations
- The quark density evolution equation:

$$\frac{d}{dt}q_i(x,t) = \frac{\alpha_s(t)}{2\pi}[q_i \bigotimes P_{qq}] + \frac{\alpha_s(t)}{2\pi}[g \bigotimes P_{qg}]$$

  with the notation:

$$[q \bigotimes P] = [P \bigotimes q] = \int_x^1 dy \frac{q(y,t)}{y} \cdot P(\frac{x}{y})$$

- Interpretation of this equation: The variation of the quark density is due to the convolution of the quark density at a higher energy times the probability of finding a quark in a quark (with the right energy fraction) plus the gluon density at a higher energy times the probability of finding a quark (of the given flavour i) in a gluon

- The gluon density evolution equation:

$$\frac{d}{dt}g(x,t) = \frac{\alpha_s(t)}{2\pi}[\sum_i (q_i + \bar{q}_i) \bigotimes P_{gq}] + \frac{\alpha_s(t)}{2\pi}[g \bigotimes P_{gg}]$$

- The explicit forms of the splitting functions can be derived from the QCD vertices. They are universal, i.e., process-independent

$$\begin{aligned}
P_{qq} &= \frac{4}{3}\left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right] + \mathcal{O}(\alpha_s) \\
P_{gq} &= \frac{4}{3}\frac{1+(1-x)^2}{x} + \mathcal{O}(\alpha_s) \\
P_{qg} &= \frac{1}{2}[x^2 + (1-x)^2] + \mathcal{O}(\alpha_s) \\
P_{gg} &= 6\left[\frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x)\right] + \frac{33-2n_f}{6}\delta(1-x) + \mathcal{O}(\alpha_s)
\end{aligned}$$

- The " + " distribution is defined as, for a generic non singular weight function $f(x)$

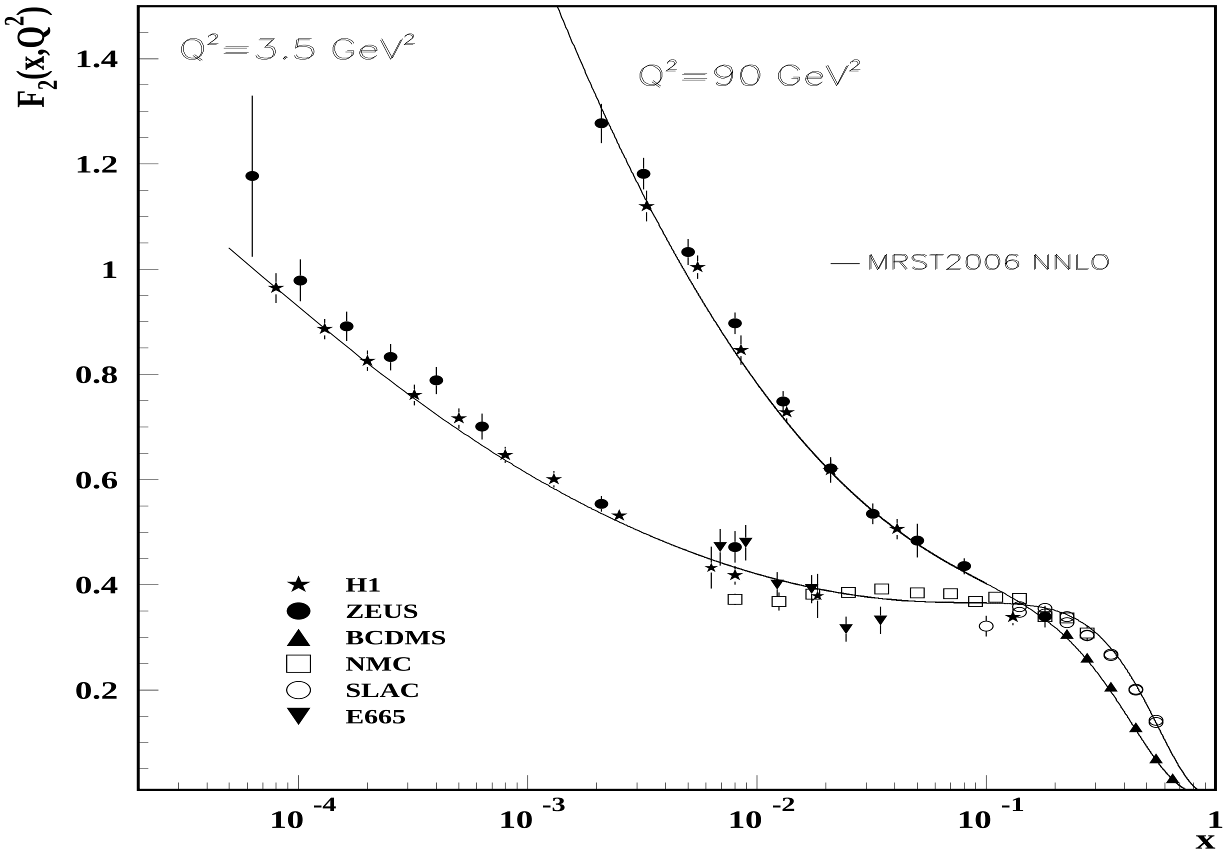
$$\int_0^1 \frac{f(x)}{(1-x)_+}dx = \int_0^1 \frac{f(x)-f(1)}{1-x}dx$$

- The splitting functions satify the normalization conditions:

$$\int_0^1 P_{qq}(x)dx = 0; \quad \int_0^1 [P_{qq}(x) + P_{gq}(x)]xdx = 0; \quad \int_0^1 [2n_f P_{qg}(x) + P_{gg}(x)]xdx = 0$$

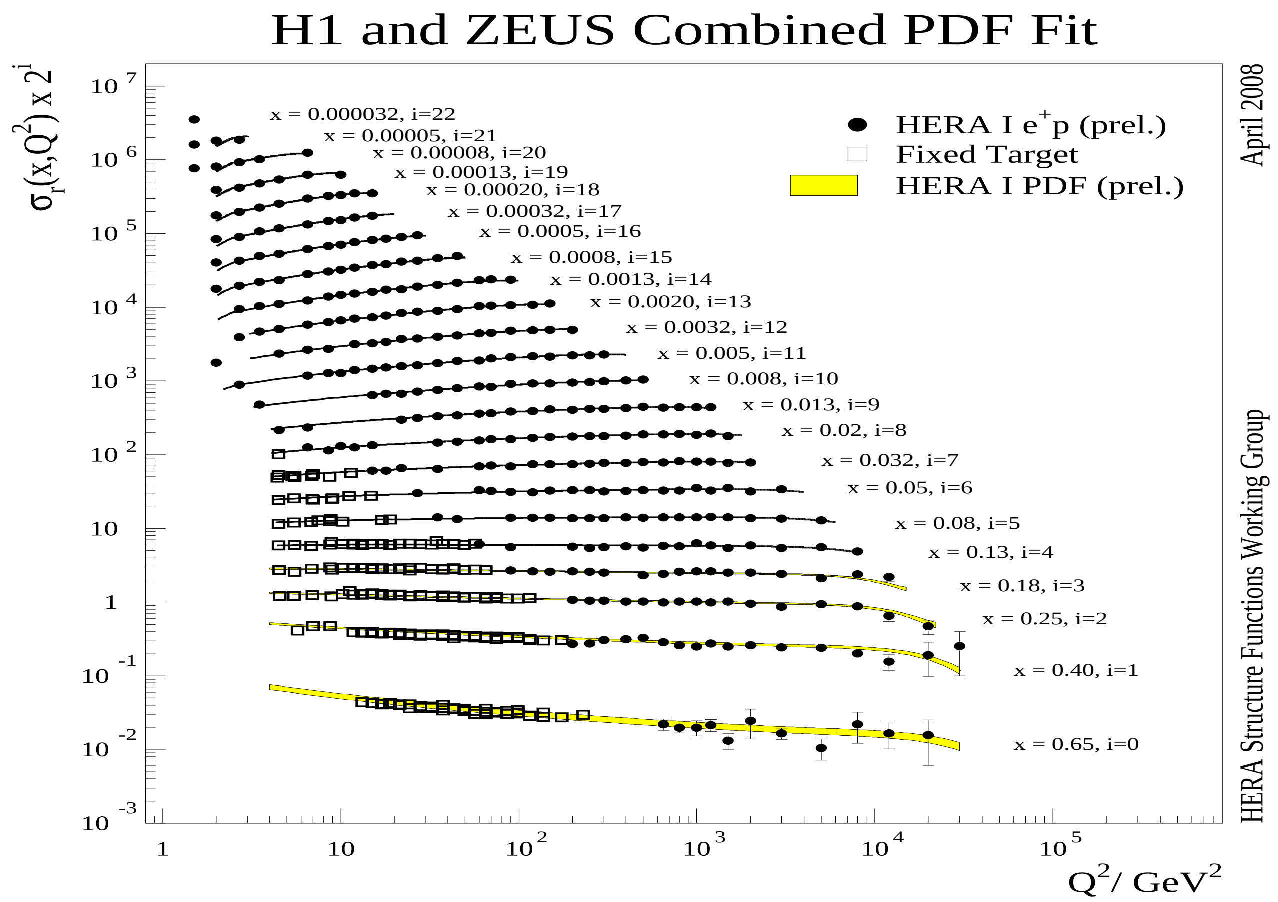
The proton structure function $F_2(x, Q^2)$

Particle Data Group 2007



Measurements of the proton SF

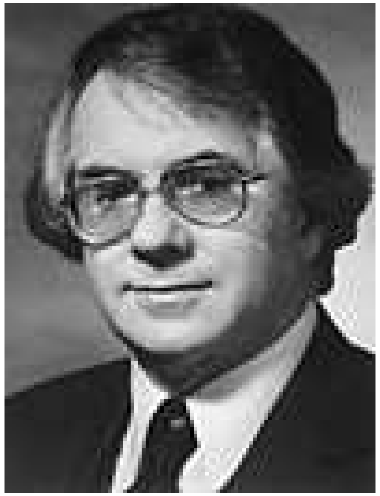
HERA Structure Function Working Group



Nobel Prize for Physics 1979

Nobelprize.org

"for their contributions to the theory of the unified weak and electromagnetic interaction between elementary particles, including, inter alia, the prediction of the weak neutral current"



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1/3 of the prize
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Harvard University, Lyman Laboratory, Cambridge, MA, USA

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International Centre for Theoretical Physics, Trieste, Italy; Imperial College London, United Kingdom

Steven Weinberg
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USA
Harvard University, Cambridge, MA, USA

Structure of the first generation of leptons and quarks in SM

$$\begin{pmatrix} u & u & u \\ d & d & d \end{pmatrix}^{L}_{1/6}$$

$$\begin{pmatrix} \nu \\ e \end{pmatrix}^{L}_{-1/2}$$

$$(\, u \quad u \quad u \,)^{R}_{2/3}$$

$$(\, d \quad d \quad d \,)^{R}_{-1/3}$$

$$(\, e \,)^{R}_{-1}$$

No ν^{R}

SU(3) x SU(2) x U(1)

mixed, not unified

- The hypercharge quantum numbers Y are fixed by the relation $Q = I_3 + Y$
- Q are the electric charges (in units of the proton charge):

$$Q\begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} +2/3 \\ -1/3 \end{pmatrix}; Q\begin{pmatrix} \nu_e \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

- I_3 is the third-component of the weak isospin:

$$I_3\begin{pmatrix} u \\ d \end{pmatrix}_{\mathrm{L}} = \begin{pmatrix} +1/2 \\ -1/2 \end{pmatrix}; I_3\begin{pmatrix} \nu_e \\ e \end{pmatrix}_{\mathrm{L}} = \begin{pmatrix} +1/2 \\ -1/2 \end{pmatrix}$$

- $I = I_3 = 0$ for $u_{\mathrm{R}}, d_{\mathrm{R}}, e_{\mathrm{R}}$

EW Lagrangian Density

Lagrangian Density of the SM Electroweak Sector

- The symmetry group of the EW theory

$$G = SU(2)_L \otimes U(1)_Y$$

- Contains four different gauge bosons $W_\mu^i(x)$ ($i = 1, 2, 3$) and $B^\mu(x)$
- For simplicity, consider a single family of Quarks and Leptons:

$$\text{Quarks:} \qquad q_L(x) = \begin{pmatrix} u(x) \\ d(x) \end{pmatrix}_L, \quad u_R(x), \quad d_R(x)$$

$$\text{Leptons:} \qquad \ell_L(x) = \begin{pmatrix} \nu_e(x) \\ e^-(x) \end{pmatrix}_L, \quad \nu_{eR}(x), \quad e_R^-(x)$$

- For quark family, Lagrangian invariant under local G transformations

$$\mathcal{L}(x) = \bar{q}_L(x)\, i\gamma^\mu D_\mu q_L(x) + \bar{u}_R(x)\, i\gamma^\mu D_\mu u_R(x) + \bar{d}_R(x)\, i\gamma^\mu D_\mu d_R(x)$$

- Quark covariant derivatives $[W_\mu(x) \equiv (\sigma_i/2)\, W_\mu^i(x)]$

$$\begin{aligned} D_\mu q_L(x) &\equiv [\partial_\mu - igW_\mu(x) - ig' y_1 B_\mu(x)]\, q_L(x) \\ D_\mu u_R(x) &\equiv [\partial_\mu - ig' y_2 B_\mu(x)]\, u_R(x) \\ D_\mu d_R(x) &\equiv [\partial_\mu - ig' y_3 B_\mu(x)]\, d_R(x) \end{aligned}$$

- σ_i ($i = 1, 2, 3$) are the Pauli matrices

EW Lagrangian Density

- Properly normalized kinetic Lagrangian

$$\mathcal{L}_{\rm kin} = -\frac{1}{4}\, B_{\mu\nu} B^{\mu\nu} - \frac{1}{2}\, \mathrm{Tr}\,[W_{\mu\nu} W^{\mu\nu}] = -\frac{1}{4}\, B_{\mu\nu} B^{\mu\nu} - \frac{1}{4}\, W^i_{\mu\nu} W_i^{\mu\nu}$$

- Includes field strength tensors

$$B_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu, \qquad W_{\mu\nu} \equiv (\sigma_i/2)\, W^i_{\mu\nu}$$
$$W^i_{\mu\nu} \equiv \partial_\mu W^i_\nu - \partial_\nu W^i_\mu - g\varepsilon^{ijk}\, W^j_\mu W^k_\nu$$

- Explicit form of the $SU(2)_L$ matrix

$$W_\mu = \frac{\sigma_i}{2}\, W^i_\mu = \frac{1}{2} \begin{pmatrix} W^3_\mu & \sqrt{2}\, W^-_\mu \\ \sqrt{2}\, W^-_\mu & -W^3_\mu \end{pmatrix}$$

- Term with W_μ in $\mathcal{L}$ gives rise to the charged-current interaction of left-handed quarks with charged boson fields $W^\pm_\mu \equiv (W^1_\mu \pm i\, W^2_\mu)\,/\sqrt{2}$

$$\mathcal{L}^{\rm quarks}_{\rm CC}(x) = -\frac{g}{2\sqrt{2}}\left\{\left[\bar{d}(x)\gamma^\mu(1-\gamma_5)u(x)\right] W^+_\mu(x) + \left[\bar{u}(x)\gamma^\mu(1-\gamma_5)d(x)\right] W^-_\mu(x)\right\}$$

- Similarly, charged-current interaction of leptons are

$$\mathcal{L}^{\rm leptons}_{\rm CC}(x) = -\frac{g}{2\sqrt{2}}\left\{\left[\bar{e}(x)\gamma^\mu(1-\gamma_5)\nu_e(x)\right] W^+_\mu(x) + \left[\bar{\nu}_e(x)\gamma^\mu(1-\gamma_5)e(x)\right] W^-_\mu(x)\right\}$$

EW Lagrangian Density

- Quarks in $\mathcal{L}$ have interactions with other two bosons W_μ^3 and B_μ
- Observed bosons Z_μ and A_μ are their linear combination

$$\begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix} = \begin{pmatrix} \cos\Theta_W & \sin\Theta_W \\ -\sin\Theta_W & \cos\Theta_W \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix}$$

- In terms of the masse igenstates, the covariant derivative becomes

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}}(W_\mu^+ T^+ + W_\mu^- T-) - i\frac{1}{\sqrt{g^2+g'^2}} Z_\mu (g^2 T^3 - g'^2 Y) - i\frac{gg'}{\sqrt{g^2+g'^2}} A_\mu (T^3 + Y)$$

- To put this expression in a more useful form, we identify the electron charge e with $e = \frac{gg'}{\sqrt{g^2+g'^2}}$, and identify the electric charge with $Q = T^3 + Y$
- Neutral-current Lagrangian can be written as

$$\mathcal{L}_{\rm NC} = \mathcal{L}_{\rm QED} + \mathcal{L}_{\rm NC}^Z$$

- First term is the usual QED Lagrangian $[g\sin\Theta_W = g'\cos\Theta_W = e]$

$$\mathcal{L}_{\rm QED} = -eA_\mu J_{\rm em}^\mu = -eA_\mu \left[Q_e\left(\bar{e}\gamma^\mu e\right) + Q_u\left(\bar{u}\gamma^\mu u\right) + Q_d\left(\bar{d}\gamma^\mu d\right)\right]$$

- Second term describes the interaction of Z-boson with neutral fermionic current

$$\mathcal{L}_{\rm NC}^{Z} = -\frac{e}{\sin(2\Theta_W)} Z_\mu J_Z^\mu = -\frac{e}{\sin(2\Theta_W)} Z_\mu \sum_f \left[\bar{f}\gamma^\mu \left(v_f - a_f \gamma_5\right) f\right]$$

- Neutral-current coupling: $a_f = T_3^f$ and $v_f = T_3^f \left(1 - 4|Q_f| \sin^2\Theta_W\right)$

f	u	d	ν_e	e
$2v_f$	$1 - \frac{8}{3}\sin^2\Theta_W$	$-1 + \frac{4}{3}\sin^2\Theta_W$	1	$-1 + 4\sin^2\Theta_W$
$2a_f$	1	-1	1	-1

Cancellation of Anomaly in the GSW Theory

- In a gauge theory of left-handed massless fermions coupled with non-Abelian gauge bosons, the axial-vector current is conserved at the tree level, but has in general an anomalous divergence due to the loop corrections. Defining the axial-vector current as $J^{\mu a} = \bar{\psi}\gamma^\mu (\frac{1-\gamma_5}{2} t^a \psi)$,

$$\langle p, \nu, b; k, \lambda, c | \partial_\mu J^{\mu a} | 0 \rangle = \frac{g^2}{8\pi^2} \epsilon^{\alpha\nu\beta\lambda} p_\alpha k_\beta . \mathcal{A}^{abc}$$

where $\mathcal{A}^{abc} = tr[t^a \{t^b, t^c\}]$

- If $\mathcal{A}^{abc} \neq 0$, the current is not conserved and we have a divergent theory. In the SM, this divergence is miraculously absent, summing over all the fermionic contributions in the loop.

EW Lagrangian Density

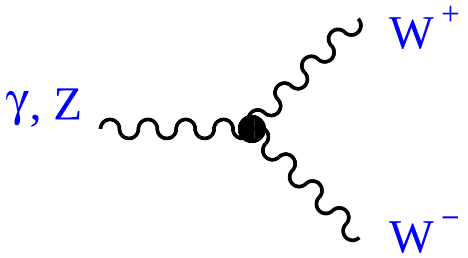
- $\mathcal{L}_{\rm kin}$ generates self-interactions among gauge bosons
- Cubic self-interactions

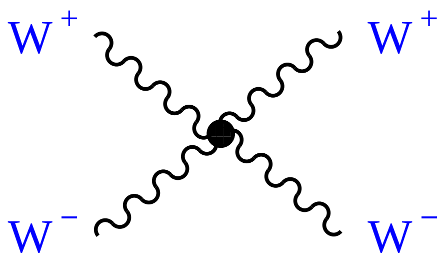
$$\begin{aligned}\mathcal{L}_{3g} &= ie\cot\Theta_W\left[W^{+\mu\nu}W^-_\mu Z_\nu - W^{-\mu\nu}W^+_\mu Z_\nu + Z^{\mu\nu}W^+_\mu W^-_\nu\right]\\ &+ ie\left[W^{+\mu\nu}W^-_\mu A_\nu - W^{-\mu\nu}W^+_\mu A_\nu + A^{\mu\nu}W^+_\mu W^-_\nu\right]\end{aligned}$$

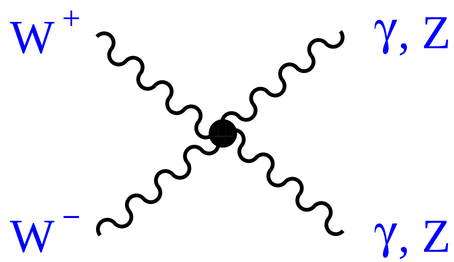
- Quartic self-interactions

$$\begin{aligned}\mathcal{L}_{4g} = & -\frac{e^2}{2\sin^2\Theta_W}\left[W^{+\mu}W^-_\mu W^{+\nu}W^-_\nu - W^{+\mu}W^+_\mu W^{-\nu}W^-_\nu\right]\\ & -e^2\cot^2\Theta_W\left[W^{+\mu}W^-_\mu Z^\nu Z_\nu - W^+_\mu Z^\mu W^-_\nu Z^\nu\right]\\ & -e^2\left[W^{+\mu}W^-_\mu A^\nu A_\nu - W^+_\mu A^\mu W^-_\nu A^\nu\right]\\ & -e^2\cot\Theta_W\left[2W^{+\mu}W^-_\mu Z^\nu A_\nu - W^+_\mu Z^\mu W^-_\nu A^\nu - W^+_\mu A^\mu W^-_\nu Z^\nu\right]\end{aligned}$$

- $\boldsymbol{\mathcal{L}_{3g}}$ and $\boldsymbol{\mathcal{L}_{4g}}$ entail the following vertices







The Higgs Mechanism

- Consider $SU(2)_L$ doublet of complex scalar fields $\phi(x) = \begin{pmatrix} \phi^{(+)}(x) \\ \phi^{(0)}(x) \end{pmatrix}$

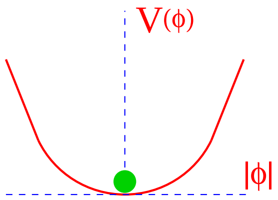
$$\mathcal{L}_S(x) = (\partial_\mu \phi)^\dagger \partial^\mu \phi - V(\phi), \quad V(\phi) = \mu^2 \phi^\dagger \phi + h \left(\phi^\dagger \phi\right)^2$$

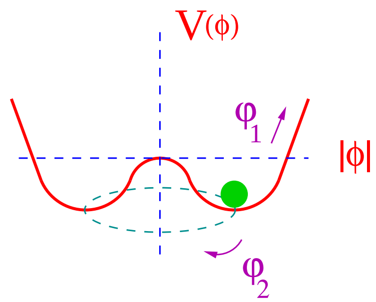
- $\boldsymbol{\mathcal{L}_S(x)}$ is invariant under *global phase transformation*:

$$\boldsymbol{\phi(x) \to \phi'(x) = e^{i\theta} \phi(x)}$$

- To get ground states, $\boldsymbol{V(\phi)}$ should be bounded from below, $\boldsymbol{h > 0}$
- For the quadratic term, there are two possibilities:

1. $\boldsymbol{\mu^2 > 0}$: trivial minimum $|\langle \mathbf{0}|\boldsymbol{\phi^{(0)}}|\mathbf{0}\rangle| = \mathbf{0}$; massive scalar particle
2. $\boldsymbol{\mu^2 < 0}$: infinite set of degenerate states with minimum energy due to the $\boldsymbol{U(1)}$ phase-invariance of $\boldsymbol{\mathcal{L}_S(x)}$
3. Choosing a particular solution, the symmetry is broken spontaneously





EW Lagrangian Density

- Now, let us consider a local gauge transformation; i.e. $\boldsymbol{\theta^i = \theta^i(x)}$
- In the physical (unitary) gauge $\theta^i(x) = 0$, only one neutral scalar field $H(x)$ (the Higgs field) remains in the model
- The gauged scalar Lagrangian of the Goldstone model (i.e., Scalar Lagrangian including interactions with gauge bosons) is:

$$\mathcal{L}_S(x) = (D_\mu \phi)^\dagger D^\mu \phi - \mu^2 \phi^\dagger \phi - h \left(\phi^\dagger \phi\right)^2 \qquad [h > 0,\, \mu^2 < 0]$$

- Covariant derivative under local $SU(2)_L \otimes U(1)_Y$ transformation

$$D_\mu \phi(x) \equiv \left[\partial_\mu + igW_\mu(x) + ig' y_\phi B_\mu(x)\right] \phi(x)$$

- Hypercharge of the scalar field $\quad y_\phi = Q_\phi - T_3 = 1/2$
- $\mathcal{L}_S(x)$ is invariant under local $\boldsymbol{SU(2) \otimes U(1)}$ transformations
- The potential is very similar to the one considered earlier. There is an infinite set of degenerate states, satisfying

$$\boldsymbol{|\langle 0|\phi^{(0)}|0\rangle| = \sqrt{\frac{-\mu^2}{2h}} \equiv \frac{v}{\sqrt{2}}}$$

- Choice of a particular ground state makes $SU(2)_L \otimes U(1)_Y$ spontaneously broken to the electromagnetic $U(1)_{\rm QED}$
- Now, let us parametrize the scalar doublet as

$$\phi(x) = \exp\left\{ i \, \frac{\sigma_i}{2} \, \theta^i(x) \right\} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}$$

- There are four real fields $\theta^i(x)$ and $H(x)$
- Local $SU(2)_L$ invariance of Lagrangian allows to rotate away any $\theta^i(x)$ dependence. They ($\theta^i(x)$) are the (would be) massless Goldstone bosons associated with the SSB mechanism
- In terms of physical fields

$$\mathcal{L}_S = \frac{1}{4} h \, v^4 + \mathcal{L}_H + \mathcal{L}_{HG^2}$$

- Pure Higgs Lagrangian has the form

$$\mathcal{L}_H = \frac{1}{2} \partial^\mu H \, \partial_\mu H - \frac{1}{2} M_H^2 \, H^2 - \frac{M_H^2}{2v} \, H^3 - \frac{M_H^2}{8v^2} \, H^4$$

- From $\mathcal{L}_{HG^2}$, one sees that the vacuum expectation value of the neutral scalar field has generated a quadratic term for the $W^\pm$ and the Z boson

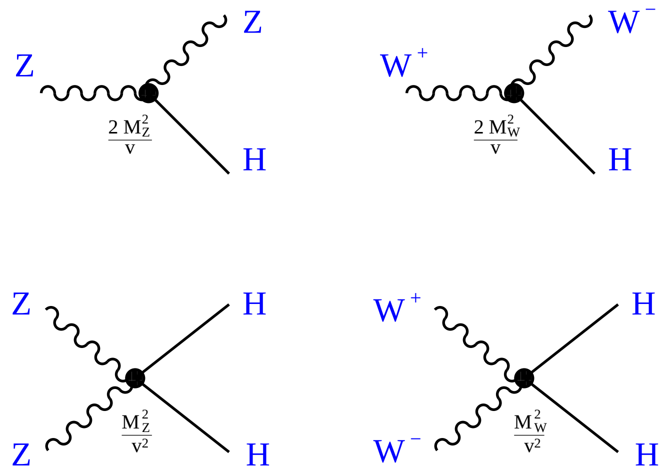
EW Lagrangian Density

- Higgs coupling to gauge bosons

$$\mathcal{L}_{HG^2} = M_W^2\, W^{+\mu} W_\mu^- \left[1 + \frac{2}{v} H + \frac{1}{v^2} H^2\right] + \frac{1}{2} M_Z^2\, Z^\mu Z_\mu \left[1 + \frac{2}{v} H + \frac{1}{v^2} H^2\right]$$

- $W^\pm$- and Z-bosons acquired masses

$$M_Z \cos\Theta_W = M_W = \frac{1}{2} g v$$



- Experimental measurements

$$M_Z = 91.1875 \pm 0.0021 \text{ GeV}$$
$$M_W = 80.398 \pm 0.025 \text{ GeV}$$

- Electroweak mixing angle

$$\sin^2\Theta_W = 1 - M_W^2/M_Z^2 = 0.223$$

- Fermi constant $G_F = 1.166371 \times 10^{-5}$ GeV^{-2} gives direct determination of the electroweak scale or the Higgs VEV

$$v = \left[\sqrt{2}\, G_F\right]^{-1/2} = 246 \text{ GeV}$$

EW Lagrangian Density

Fermion Mass Generation

- Fermionic mass term $\mathcal{L}_m = -m\bar{\psi}\psi = -m\left[\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L\right]$
  is not allowed as it breaks gauge symmetry
- Gauge-invariant fermion-scalar coupling, or Yukawa couplings, are:

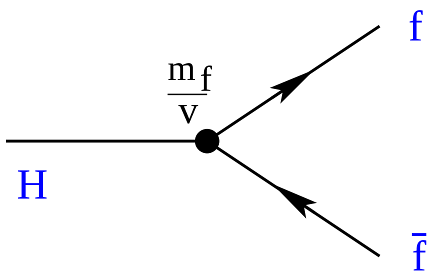
$$\begin{aligned}\mathcal{L}_Y &= -C_1\left(\bar{u},\,\bar{d}\right)_L \begin{pmatrix}\phi^{(+)}\\ \phi^{(0)}\end{pmatrix} d_R - C_2\left(\bar{u},\,\bar{d}\right)_L \begin{pmatrix}\phi^{(0)*}\\ -\phi^{(-)}\end{pmatrix} u_R\\ &\quad -C_3\left(\bar{\nu}_e,\,\bar{e}\right)_L \begin{pmatrix}\phi^{(+)}\\ \phi^{(0)}\end{pmatrix} e_R + \mathrm{h.c}\end{aligned}$$

- Second term involves a charged-conjugate scalar field $\phi^C \equiv i\sigma_2\phi^*$
- In physical (unitary) gauge after symmetry breaking, $\mathcal{L}_Y$ simplifies

$$\mathcal{L}_Y(x) = -\frac{1}{\sqrt{2}}\left[v + H(x)\right]\left\{C_1\,\bar{d}(x)d(x) + C_2\,\bar{u}(x)u(x) + C_3\,\bar{e}(x)e(x)\right\}$$

- VEV generates masses of the fermions
  $m_d = C_1 v/\sqrt{2}$, $m_u = C_2 v/\sqrt{2}$, etc.
- Yukawa coupling in terms of fermion masses

$$\mathcal{L}_Y = -\left[1 + \frac{H}{v}\right]\left(m_u\,\bar{u}u + m_d\,\bar{d}d + m_e\,\bar{e}e\right)$$



Flavor Dynamics

- Experimentally known that there are 6 quark flavors $u,\ d,\ s,\ c,\ b,\ t$, 3 charged leptons $e,\ \mu,\ \tau$ and 3 neutrinos $\nu_e,\ \nu_\mu,\ \nu_\tau$
- Can be organized into 3 families of quarks and leptons
- Exist 3 nearly identical copies of the same $SU(2)_L \otimes U(1)_Y$ structure; with varying particles masses
- Consider N_G generations of fermions $\nu'_j,\ \ell'_j,\ u'_j,\ d'_j$ $(j = 1, 2, \ldots, N_G)$
- The most general Yukawa-type Lagrangian has the form

$$\begin{aligned}\mathcal{L}_Y &= -\sum_{jk}\Big\{\left(\bar{u}'_j,\, \bar{d}'_j\right)_L C^{(d)}_{jk}\begin{pmatrix}\phi^{(+)}\\ \phi^{(0)}\end{pmatrix} d'_{kR} + \left(\bar{u}'_j,\, \bar{d}'_j\right)_L C^{(u)}_{jk}\begin{pmatrix}\phi^{(0)*}\\ -\phi^{(-)}\end{pmatrix} u'_{kR} \\ &\quad + \left(\bar{\nu}'_j,\, \bar{e}'_j\right)_L C^{(\ell)}_{jk}\begin{pmatrix}\phi^{(+)}\\ \phi^{(0)}\end{pmatrix} \ell'_{kR}\Big\} + \mathrm{h.c}\end{aligned}$$

- After spontaneous symmetry breaking, in unitary gauge

$$\mathcal{L}_Y = -\left[1 + \frac{H}{v}\right]\left(\bar{\mathbf{u}}'_L \mathbf{M}'_u \mathbf{u}'_R + \bar{\mathbf{d}}'_L \mathbf{M}'_d \mathbf{d}'_R + \bar{\mathbf{l}}'_L \mathbf{M}'_l \mathbf{l}'_R + \mathrm{h.c}\right)$$

- Mass matrices are defined by $\quad (\mathbf{M}'_f)_{jk} \equiv C^{(f)}_{jk}\, v/\sqrt{2}$

EW Lagrangian Density

- Mass matrices can be decomposed $(\mathbf{M}'_f)_{jk} = \mathbf{S}_f^\dagger \mathcal{M}_f \mathbf{S}_f \mathbf{U}_f$, where $\mathbf{S}_f$ and $\mathbf{U}_f$ are unitary and $\mathcal{M}_f$ is diagonal, Hermitian and positive definite
- In terms of $\mathcal{M}_u = \mathrm{diag}\,(m_u,\ m_c,\ m_t, \ldots)$, etc

$$\mathcal{L}_Y = -\left[1 + \frac{H}{v}\right]\left(\bar{\mathbf{u}}\mathcal{M}_u\mathbf{u} + \bar{\mathbf{d}}\mathcal{M}_d\mathbf{d} + \bar{\mathbf{l}}\mathcal{M}_l\mathbf{l}\right)$$

- Mass eigenstates are defined by $\mathbf{f}_L \equiv \mathbf{S}_f \mathbf{f}'_L$ and $\mathbf{f}_R \equiv \mathbf{S}_f \mathbf{U}_f \mathbf{f}'_R$
- Since $\bar{\mathbf{f}}'_L \mathbf{f}'_L = \bar{\mathbf{f}}_L \mathbf{f}_L$ and $\bar{\mathbf{f}}'_R \mathbf{f}'_R = \bar{\mathbf{f}}_R \mathbf{f}_R$, there are no flavor-changing neutral currents in the SM [Glashow-Illiopolous-Miani (GIM) mechanism]
- For charged quark current: $\bar{\mathbf{u}}'_L \mathbf{d}'_L = \bar{\mathbf{u}}_L \mathbf{S}_u \mathbf{S}_d^\dagger \mathbf{d}_L \equiv \bar{\mathbf{u}}_L \mathbf{V} \mathbf{d}_L$
- The Cabibbo-Kobayashi-Maskawa (CKM) matrix $\mathbf{V}$ is the unitary (3×3) matrix; couples any “up-type” quark with all “down-type”
- For charged lepton current: $\bar{\nu}'_L \mathbf{l}'_L = \bar{\nu}_L \mathbf{S}_l^\dagger \mathbf{l}_L \equiv \bar{\nu}_L \mathbf{l}_L$
- Charged-current Lagrangian density

$$\mathcal{L}_{\mathrm{CC}} = -\frac{g}{2\sqrt{2}}\left\{W_\mu^- \left[\sum_{ij} \bar{u}_i \gamma^\mu (1-\gamma_5) \mathbf{V}_{ij} d_j + \sum_\ell \bar{\nu}_\ell \gamma^\mu (1-\gamma_5)\ell\right] + \mathrm{h.c.}\right\}$$

EW Lagrangian Density

- General $N_G \times N_G$ unitary matrix is characterized by N_G^2 real parameters: $N_G(N_G-1)/2$ moduli and $N_G(N_G+1)/2$ phases
- Under the phase redefinitions $u_i \to \mathrm{e}^{i\phi_i}\, u_i$ and $d_j \to \mathrm{e}^{i\theta_j}\, d_j$, the mixing matrix changes as $\mathbf{V}_{ij} \to \mathbf{V}_{ij}\mathrm{e}^{i(\theta_j-\phi_i)}$; $2N_G-1$ phases are unobservable
- The number of physical free parameters in $\mathbf{V}$ then gets reduced to $(N_G-1)^2$: $N_G(N_G-1)/2$ moduli and $(N_G-1)(N_G-2)/2$ phases
- Simplest case of two generations $N_G = 2$ [1 moduli and no phases]

$$\mathbf{V} = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix}$$

- In case of three generations $N_G = 3$, there are 3 moduli and 1 phase

$$\mathbf{V} = \begin{bmatrix} c_{12}\, c_{13} & s_{12}\, c_{13} & s_{13}\, \mathrm{e}^{-i\delta_{13}} \\ -s_{12}\, c_{23} - c_{12}\, s_{23}\, s_{13}\, \mathrm{e}^{i\delta_{13}} & c_{12}\, c_{23} - s_{12}\, s_{23}\, s_{13}\, \mathrm{e}^{i\delta_{13}} & s_{23}\, c_{13} \\ s_{12}\, s_{23} - c_{12}\, c_{23}\, s_{13}\, \mathrm{e}^{i\delta_{13}} & -c_{12}\, s_{23} - s_{12}\, c_{23}\, s_{13}\, \mathrm{e}^{i\delta_{13}} & c_{23}\, c_{13} \end{bmatrix}$$

- Here $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$, with i and j being "generation" labels
- The only complex phase in the SM Lagrangian is δ_{13}; only possible source of CP-violation phenomena in the SM

Cabibbo, Kobayashi and Maskawa

The Cabibbo-Kobayashi-Maskawa Matrix

$$V_{\rm CKM} \equiv \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

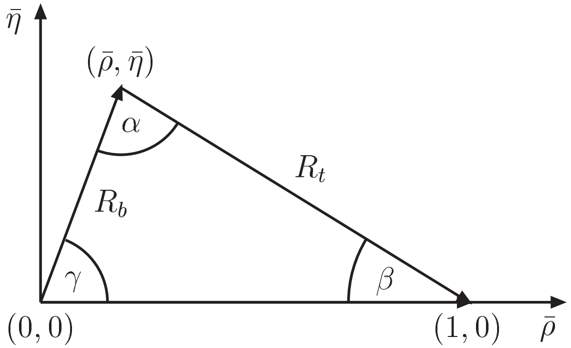
- Customary to use the handy Wolfenstein parametrization

$$V_{\rm CKM} \simeq \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda(1+iA^2\lambda^4\eta) & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2(1+i\lambda^2\eta) & 1 \end{pmatrix}$$

- Four parameters: A, λ, ρ, η
- Perturbatively improved version of this parametrization

$$\bar{\rho} = \rho(1-\lambda^2/2), \quad \bar{\eta} = \eta(1-\lambda^2/2)$$

- The CKM-Unitarity triangle $[\phi_1 = \beta; \ \phi_2 = \alpha; \ \phi_3 = \gamma]$



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DESY, Hamburg

Phases and sides of the UT

$$\alpha \equiv \arg\left(-\frac{V_{tb}^* V_{td}}{V_{ub}^* V_{ud}}\right), \qquad \beta \equiv \arg\left(-\frac{V_{cb}^* V_{cd}}{V_{tb}^* V_{td}}\right), \qquad \gamma \equiv \arg\left(-\frac{V_{ub}^* V_{ud}}{V_{cb}^* V_{cd}}\right)$$

- β and γ have simple interpretation

$$V_{td} = |V_{td}| e^{-i\beta}, \qquad V_{ub} = |V_{ub}| e^{-i\gamma}$$

- α defined by the relation: $\alpha = \pi - \beta - \gamma$
- The Unitarity Triangle (UT) is defined by:

$$R_b \mathrm{e}^{i\gamma} + R_t \mathrm{e}^{-i\beta} = 1$$

$$R_b \equiv \frac{|V_{ub}^* V_{ud}|}{|V_{cb}^* V_{cd}|} = \sqrt{\bar{\rho}^2 + \bar{\eta}^2} = \left(1 - \frac{\lambda^2}{2}\right) \frac{1}{\lambda} \left|\frac{V_{ub}}{V_{cb}}\right|$$

$$R_t \equiv \frac{|V_{tb}^* V_{td}|}{|V_{cb}^* V_{cd}|} = \sqrt{(1-\bar{\rho})^2 + \bar{\eta}^2} = \frac{1}{\lambda} \left|\frac{V_{td}}{V_{cb}}\right|$$

Current determination of $V_{\rm CKM}$

[Particle Data Group, 2008]

- $|V_{ud}| = 0.97418(27)$
- $|V_{us}| = 0.2255(19)$
- $|V_{ub}| = (3.93 \pm 0.36) \times 10^{-3}$

Unitarity of the Ist Row of $V_{\rm CKM}$

$$1 - (|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2) = (1 \pm 1) \times 10^{-3}$$

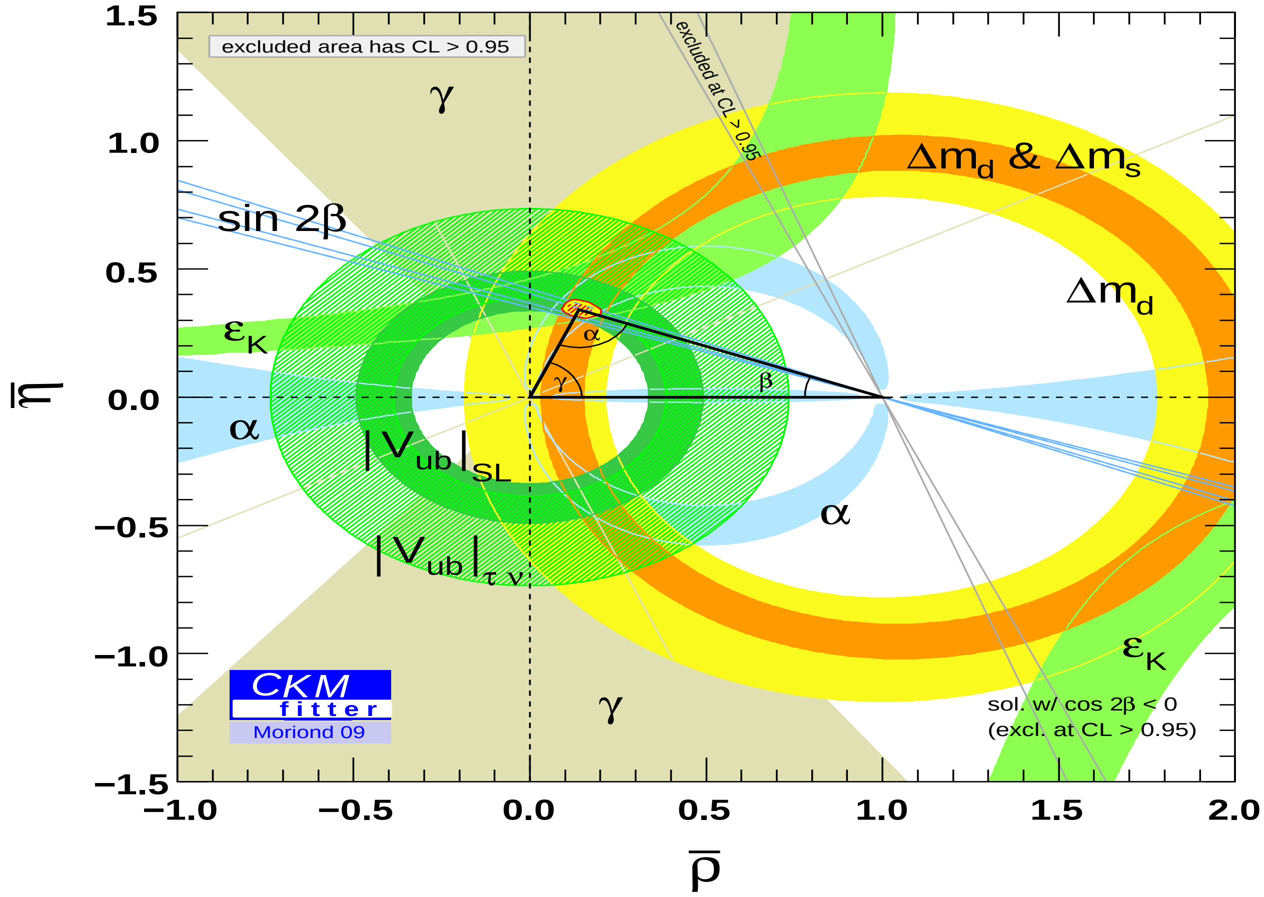
- $|V_{cd}| = 0.230(11)$
- $|V_{cs}| = 1.04 \pm 0.06$
- $|V_{cb}| = (41.2 \pm 1.1) \times 10^{-3}$

Unitarity of the second row of $V_{\rm CKM}$

$$1 - (|V_{cd}|^2 + |V_{cs}|^2 + |V_{cb}|^2) = 0.136 \pm 0.125$$

- $|V_{td}| = (8.1 \pm 2.6) \times 10^{-3}$
- $|V_{ts}| = (38.7 \pm 0.3) \times 10^{-3}$
- $|V_{tb}| > 0.74 \ (95\% C.L.)$
  - Conclusion: No Beyond-the-SM Physics in the $V_{\rm CKM}$

Current Status of the CKM-Unitarity Triangle



Precision Tests of the Electroweak Theory: Z PHYSICS

Z boson: mixture of $SU(2)$ and $U(1)$ gauge fields:

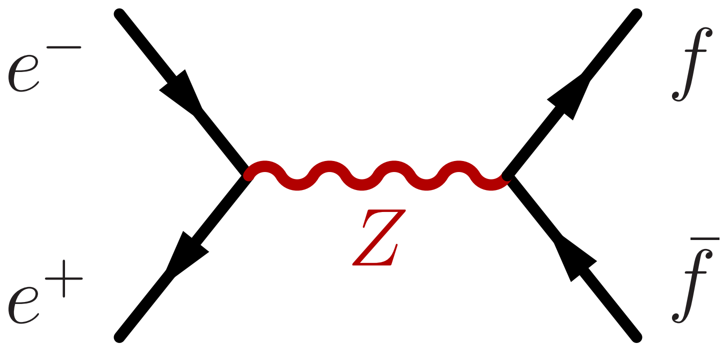
$$Z = -B\,\sin\theta_W + W^3\,\cos\theta_W \quad \text{with} \quad \tan\theta_W = g'/g$$

Z – matter interactions:

$$\mathcal{L} \; = - \tfrac{g}{\cos\theta}\,\Sigma\,\bar{f}\,\left[g_V^f\,\gamma_\mu - g_A^f\,\gamma_\mu\gamma_5\right]\,f\cdot Z_\mu$$

$$\begin{cases} g_V^f = I_{3L}^f - 2\,Q^f\sin^2\theta_W \\ g_A^f = I_{3L}^f \end{cases} \quad \underline{\text{Rad Cor:}} \quad \begin{array}{l} \text{current } \times \rho_f^{1/2} \\ \sin^2\theta_{\rm eff}^f \end{array}$$

PRODUCTION AT LEP1:



tot cx: $\sigma = \frac{12\pi}{M_Z^2}\,\frac{\Gamma_e\Gamma_f}{\Gamma_Z^2} \times \text{BW}$

FB asym: $A_{FB}^f = \frac{3}{4}\,A^e A^f$

τ polarization $\mathcal{P}^\tau$

– partial widths : $\Gamma(Z \to f\,\bar{f}) = \frac{\mathcal{C}\,G_F M_Z^3}{6\sqrt{2}\pi}\,\left[g_V^2 + g_A^2\right]$

– polarization parm.: $A^f = \frac{2 g_V\, g_A}{g_V^2 + g_A^2}$

RESULTS

$$\left.\begin{array}{ll} \underline{\text{lineshape}}: & \text{mass} \\ & \text{width} \\ \underline{\text{couplings}}: & \text{branching ratios} \\ & \text{asymmetries} \end{array}\right\}$$

$$M_Z = 91\,187.5 \pm 2.1 \text{ MeV}$$
$$\Gamma_Z = \ \ 2\,495.2 \pm 2.3 \text{ MeV}$$
$$\sin^2\theta^l_{\text{eff}} = 0.2324 \pm 0.0012$$

$\Leftarrow$ experiment
$\oplus$ theory

per-mille accuracies: <u>sensitivity to quantum effects</u>
$[\text{mod. } A^b_{FB} \ \& \ A_{LR}]$ <u>probing new high scales</u>

IMPORTANT CONSEQUENCES

within / beyond SM: # ν's (light)
top-quark prediction
unification of gauge couplings

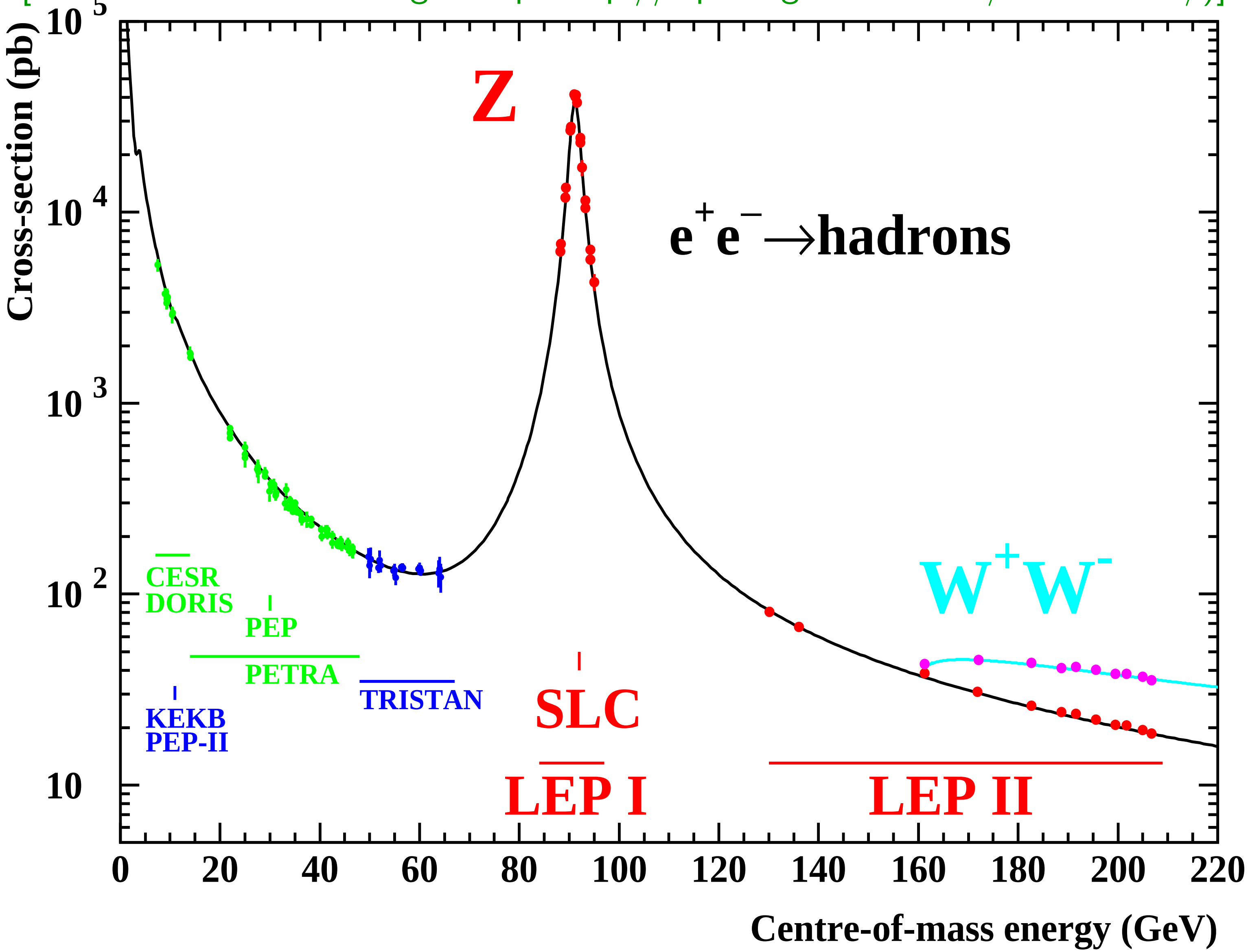
1.) NUMBER OF LIGHT NEUTRINOS

light SM-type ν's: $\Gamma_Z = \Gamma_{vis} + N_\nu \cdot \Gamma_{\nu\bar{\nu}}$

$$\underline{N_\nu = 2.985 \pm 0.008}$$

Cross-section $e^+e^- \to$ hadrons at LEP

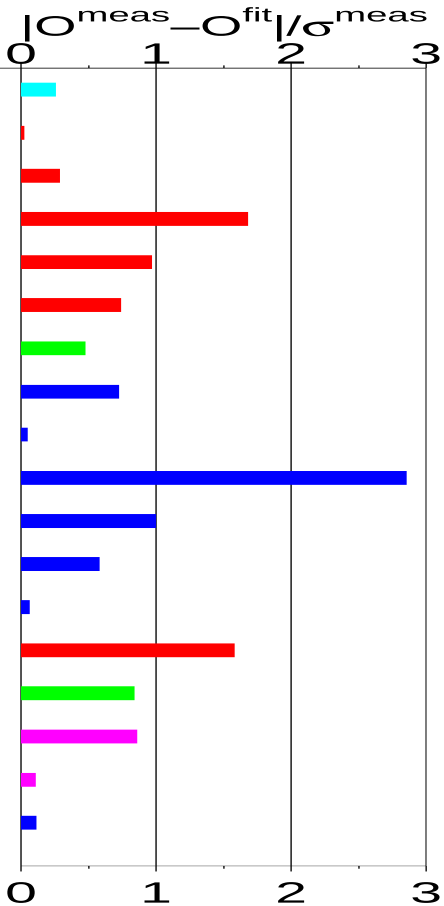
[LEP Electroweak Working Group: http://lepewwg.web.cern.ch/LEPEWWG/)]



Pull on the SM from Precision Electroweak Measurements

[LEP Electroweak Working Group: http://lepewwg.web.cern.ch/LEPEWWG/)]

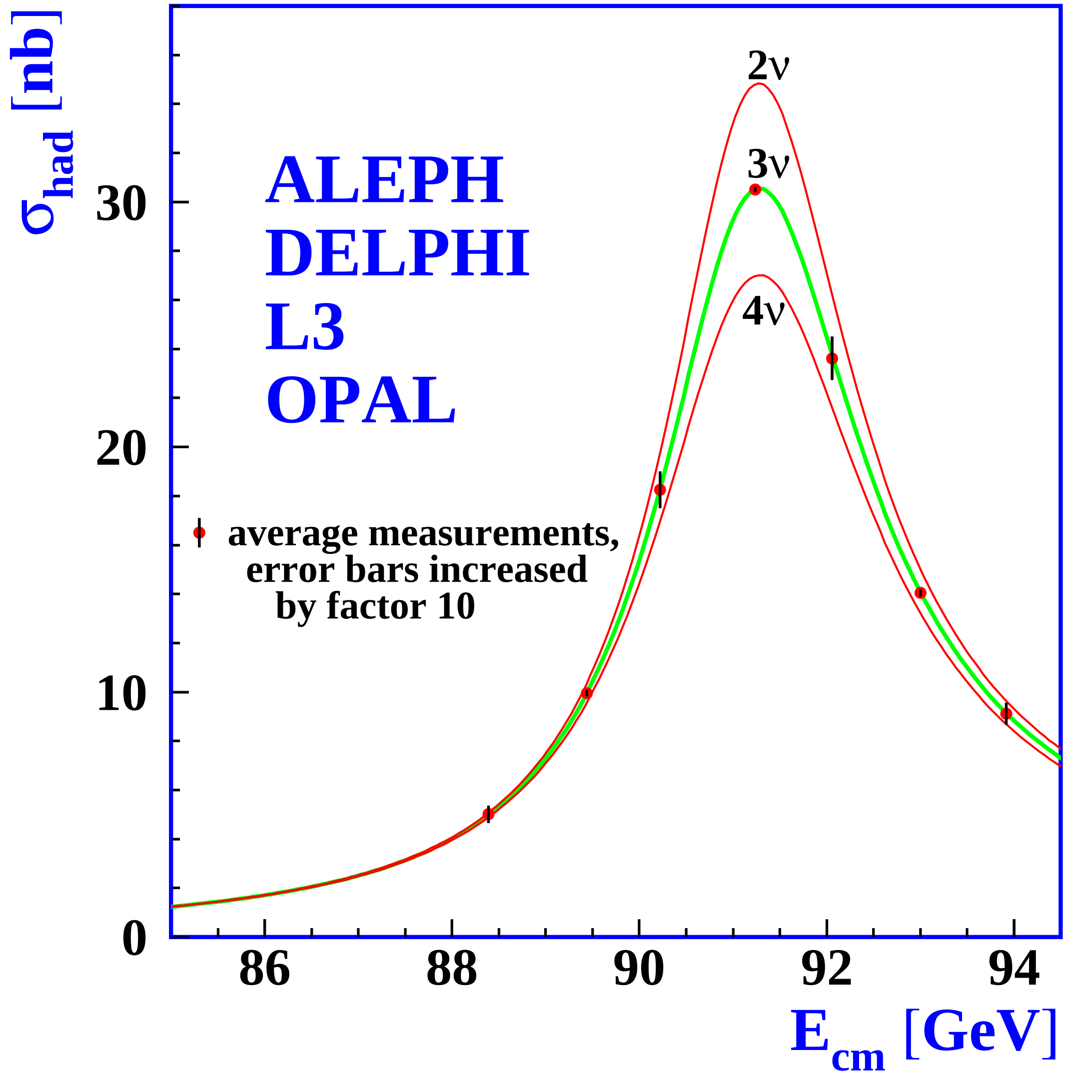
	Measurement	Fit
$\Delta\alpha^{(5)}_{had}(m_Z)$	0.02758 ± 0.00035	0.02767
m_Z [GeV]	91.1875 ± 0.0021	91.1874
Γ_Z [GeV]	2.4952 ± 0.0023	2.4959
σ^0_{had} [nb]	41.540 ± 0.037	41.478
R_l	20.767 ± 0.025	20.743
$A^{0,l}_{fb}$	0.01714 ± 0.00095	0.01643
$A_l(P_\tau)$	0.1465 ± 0.0032	0.1480
R_b	0.21629 ± 0.00066	0.21581
R_c	0.1721 ± 0.0030	0.1722
$A^{0,b}_{fb}$	0.0992 ± 0.0016	0.1038
$A^{0,c}_{fb}$	0.0707 ± 0.0035	0.0742
A_b	0.923 ± 0.020	0.935
A_c	0.670 ± 0.027	0.668
A_l(SLD)	0.1513 ± 0.0021	0.1480
$\sin^2\theta^{lept}_{eff}(Q_{fb})$	0.2324 ± 0.0012	0.2314
m_W [GeV]	80.398 ± 0.025	80.377
Γ_W [GeV]	2.097 ± 0.048	2.092
m_t [GeV]	172.6 ± 1.4	172.8



March 2008

Determination of the number of Neutrinos at LEP

[LEP Electroweak Working Group: http://lepewwg.web.cern.ch/LEPEWWG/)]

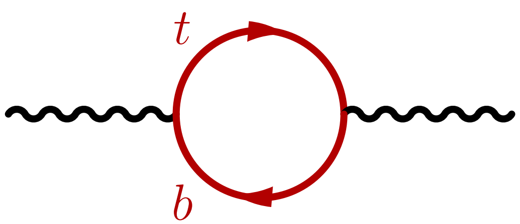


Important Consequences from LEP Physics

2.) TOP QUARK:

(i) existence: $I^b_{3L} = -\frac{1}{2}$ and $I^b_{3R} = 0 \;\; \Rightarrow ; \quad t =$ isospin-partner to b

(ii) mass:



$M_Z, \sin^2\theta^l_{eff}, G_F \;\; \rightarrow$

$$\left.\begin{array}{l} \rho = 1 + \Delta\rho_t + \Delta\rho_H \\ \Delta\,\rho_t \sim G_F m_t^2 \end{array}\right\}$$

LEP1/SLD: $m_t = 172.6^{+13}_{-10}$ GeV

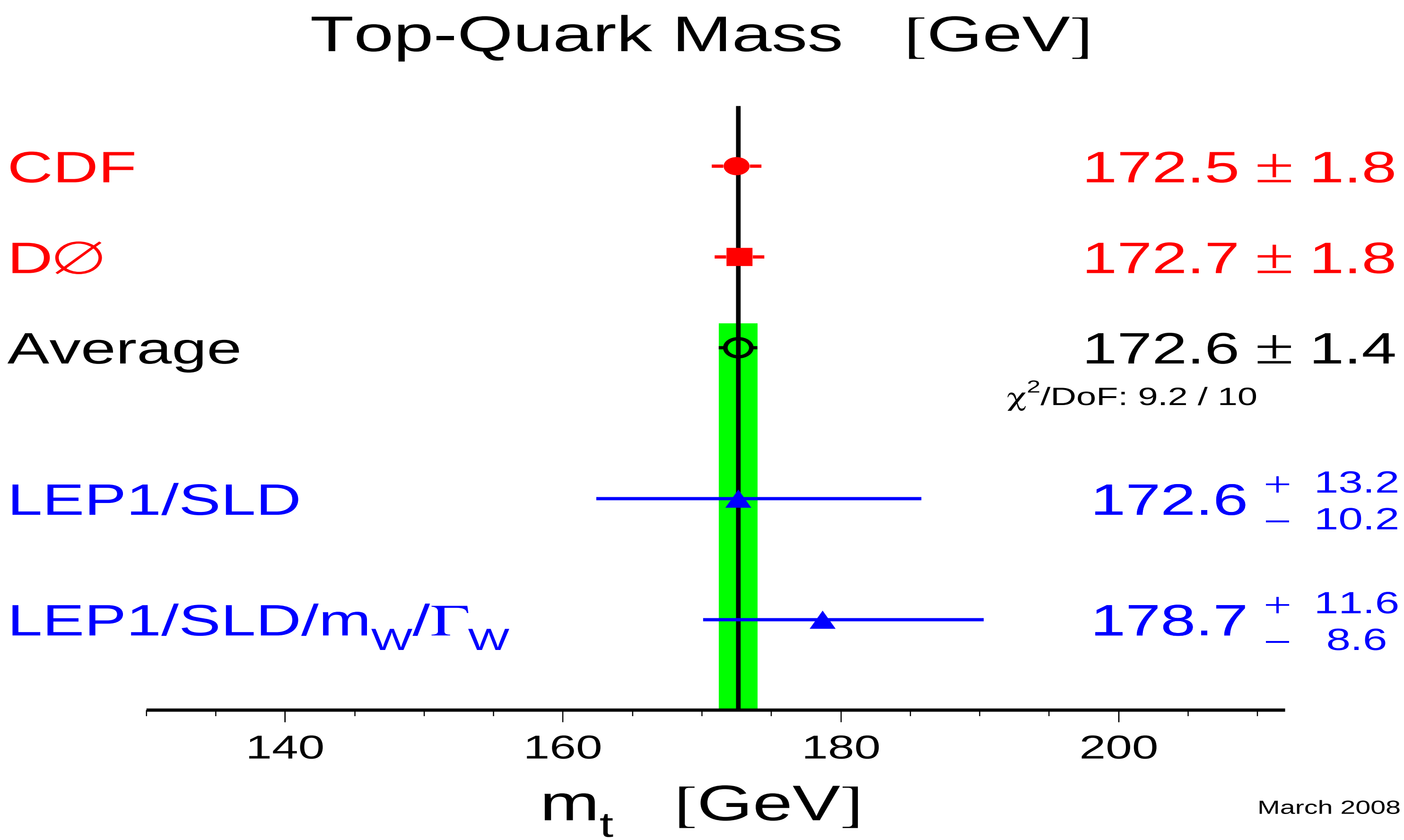
Tevatron: $m_t = 172.6 \pm 1.4$ GeV

prediction of t mass: $[1994: \;\; 173 \pm 26\,\mathrm{GeV}]$

successful SM confirmation at quantum level

Current Measurements of M_t at the Tevatron and LEP/SLD

[From the LEP Electroweak Working Group: http://lepewwg.web.cern.ch/LEPEWWG/]



- In agreement with the indirect measurements through loop effects at LEP1 and SLD:

$$M_t(\mathbf{LEP1/SLD}) = \mathbf{172.6^{+13.2}_{-10.2}}\ \text{GeV}$$

Nobel Prize for Physics 1999

"for elucidating the quantum structure of electroweak interactions in physics"

Gerardus 't Hooft
1/2 of the prize
the Netherlands
Utrecht University

Martinus J.G. Veltman
1/2 of the prize
the Netherlands
Bilthoven, the Netherlands

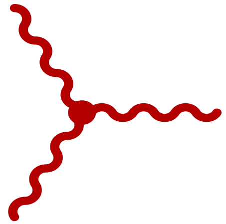
Precision Tests of the Electroweak Theory: W PHYSICS

Central Issues:

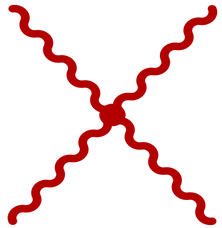
(i) Static properties of $W^{\pm}$ bosons: mass, width, ...

(ii) Yang-Mills gauge field dynamics: $\mathcal{L} = -\frac{1}{2}\langle \vec{W}^2_{\mu\nu}\rangle \ldots$

tri-linear couplings:

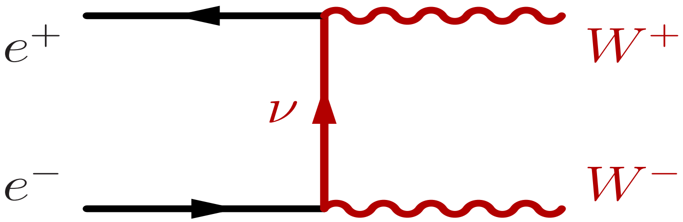


quattro-linear couplings:

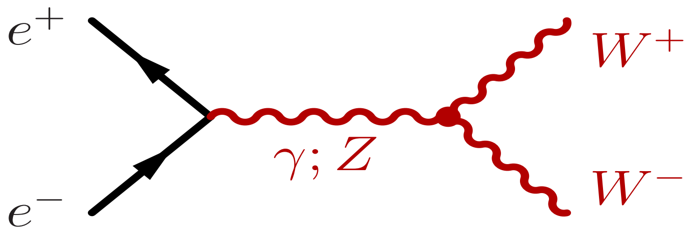


• Establish Weyl & Yang / Mills gauge principle as fundamental base for [SM] forces

PRODUCTION:

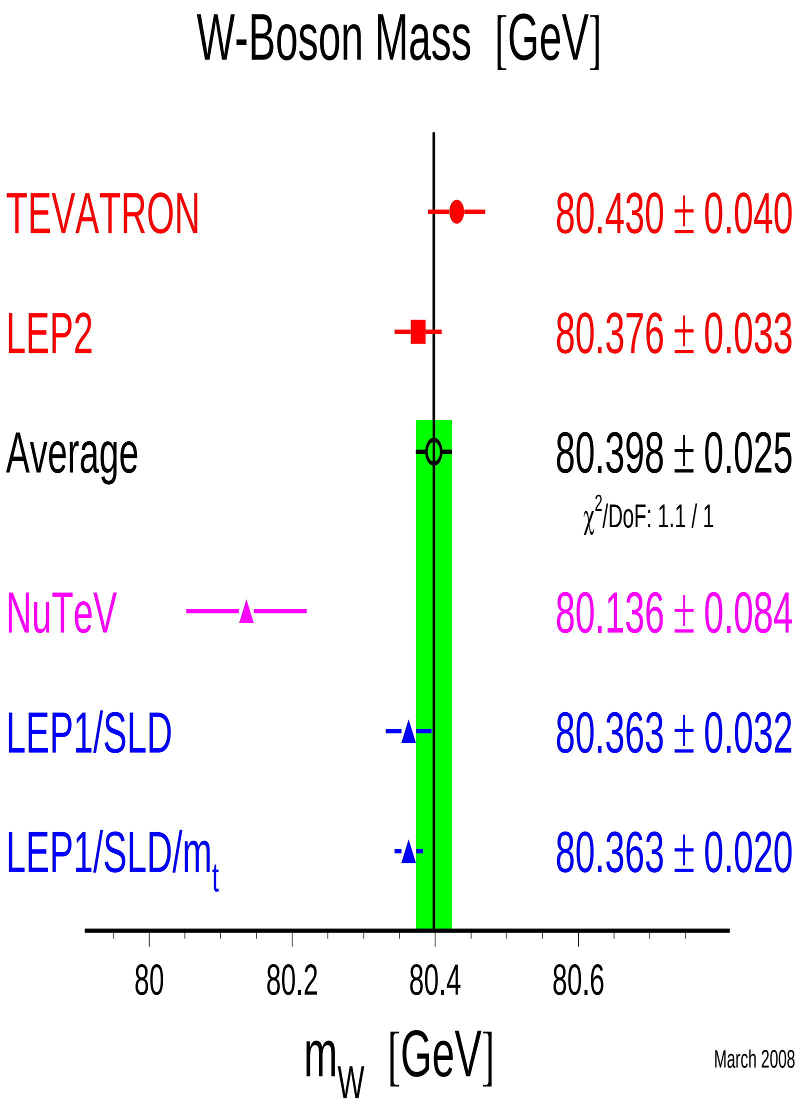


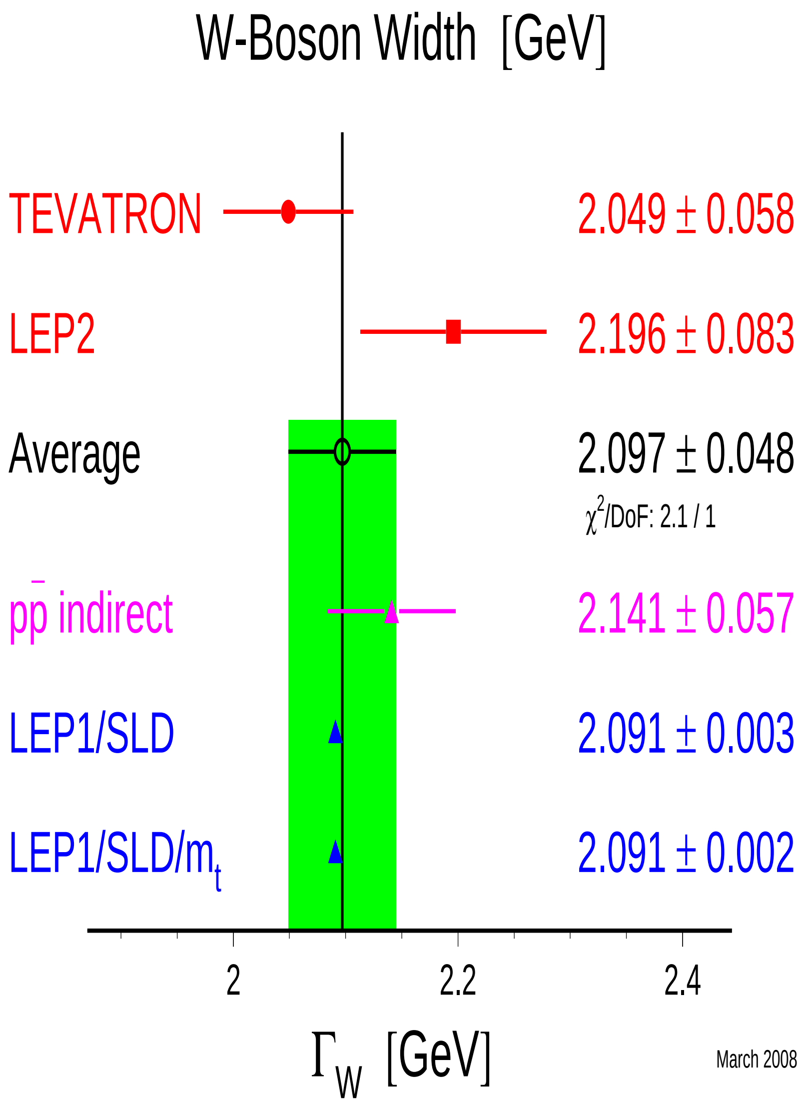
$\oplus$



$$M_W = 80.398 \pm 0.025 \text{ GeV}$$
$$\Gamma_W = 2.097 \pm 0.048 \text{ GeV}$$

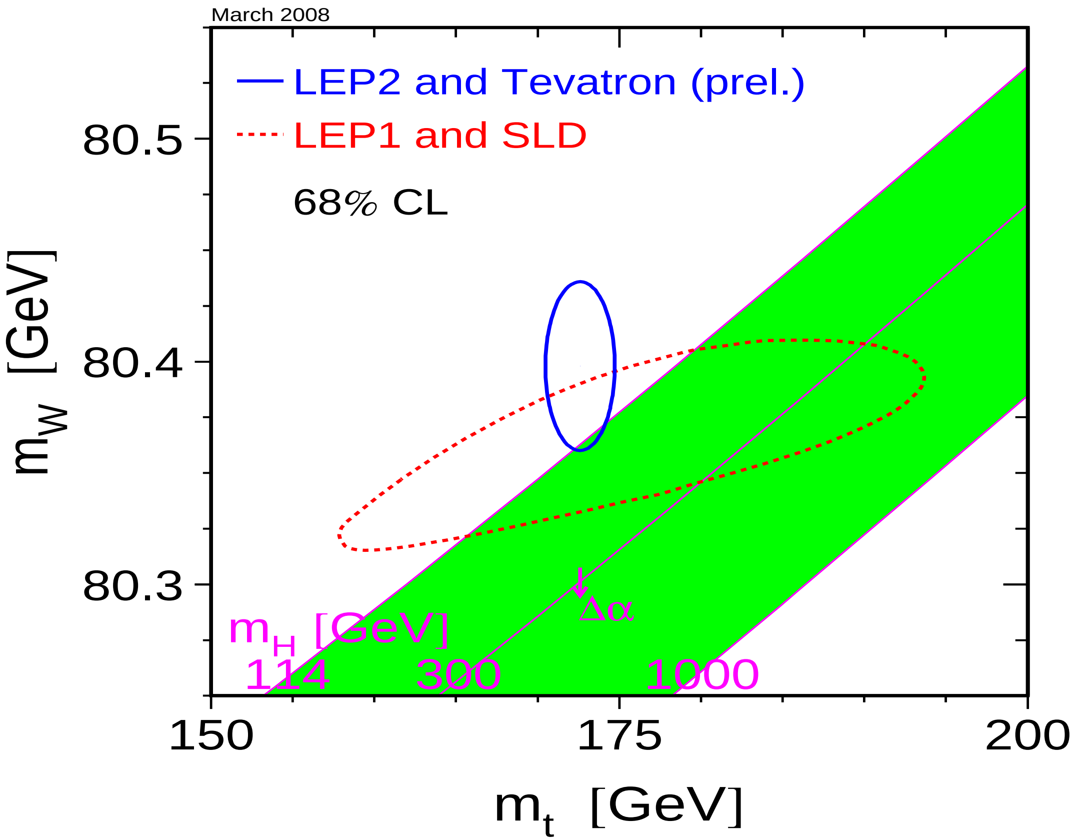
Measurement of M_W and Γ_W





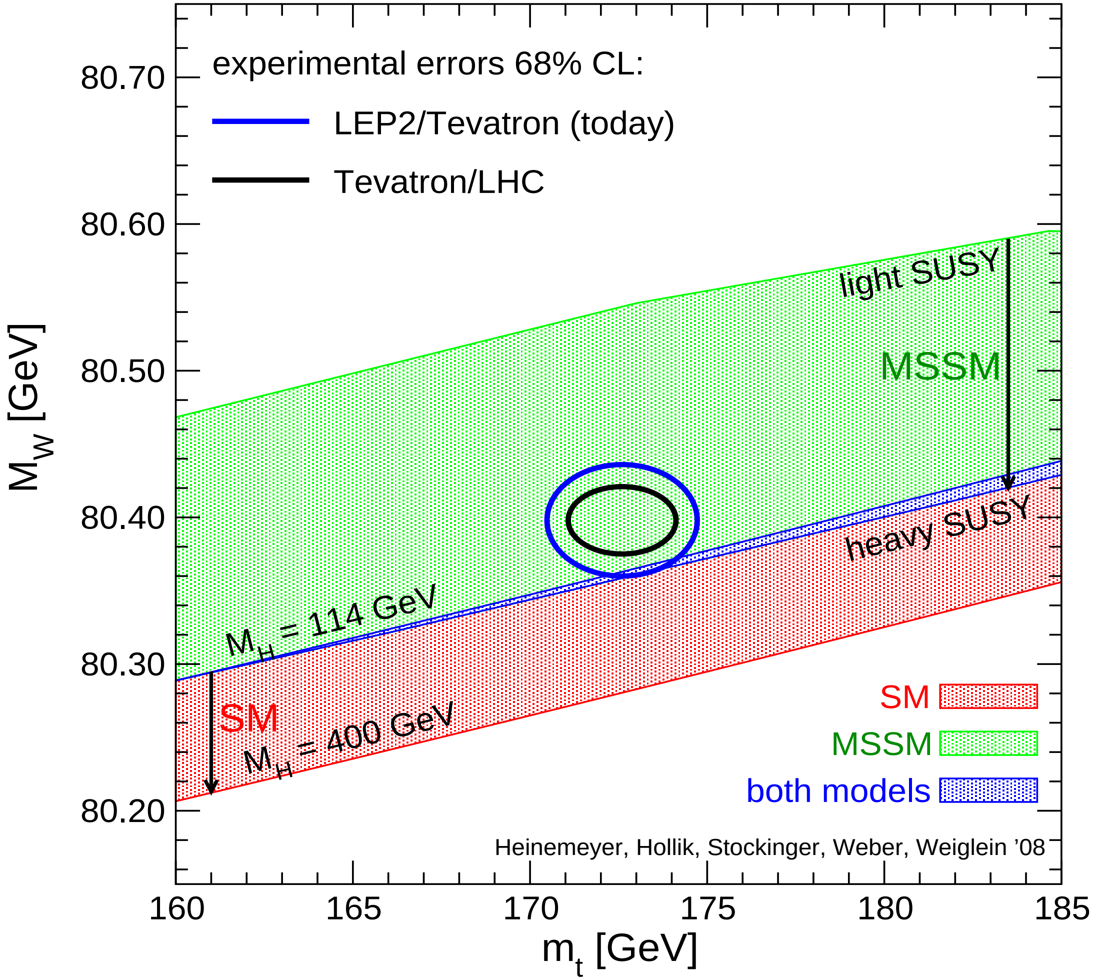
Relationship between M_W and M_t in the SM

[LEP Electroweak working group: http://lepewwg.web.cern.ch/LEPEWWG/]



Relationship between M_W and M_t in the SM and MSSM

[Heinemeyer et al. hep-ph/0604147; Updated by Heinemeyer (2008)]

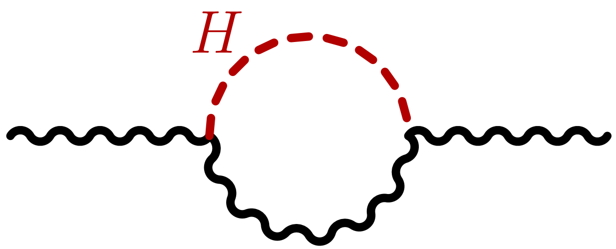


• Fit to the precision electroweak data in the SM & MSSM. The curve labelled heavy SUSY assumes the supersymmetric parameters are set at 2 TeV

Current Searches of the Higgs Boson

Approaching fourth step:

1.) VIRTUAL HIGGS BOSON:



W, Z properties:
$\rho = 1 + \Delta\rho_t + \Delta\rho_H$
$\Delta\rho_H \sim G_F M_Z^2 \log M_H^2$

$\boxed{M_H = 87^{+36}_{-27} \text{ GeV and } M_H < 190 \text{ GeV [95\% CL]}}$

[fit probability $\sim$ 36%]

2.) DIRECT SEARCH: $\boxed{M_H \geq 114.4 \text{ GeV}}$

[1.7σ excess for $\geq$ 115 GeV]

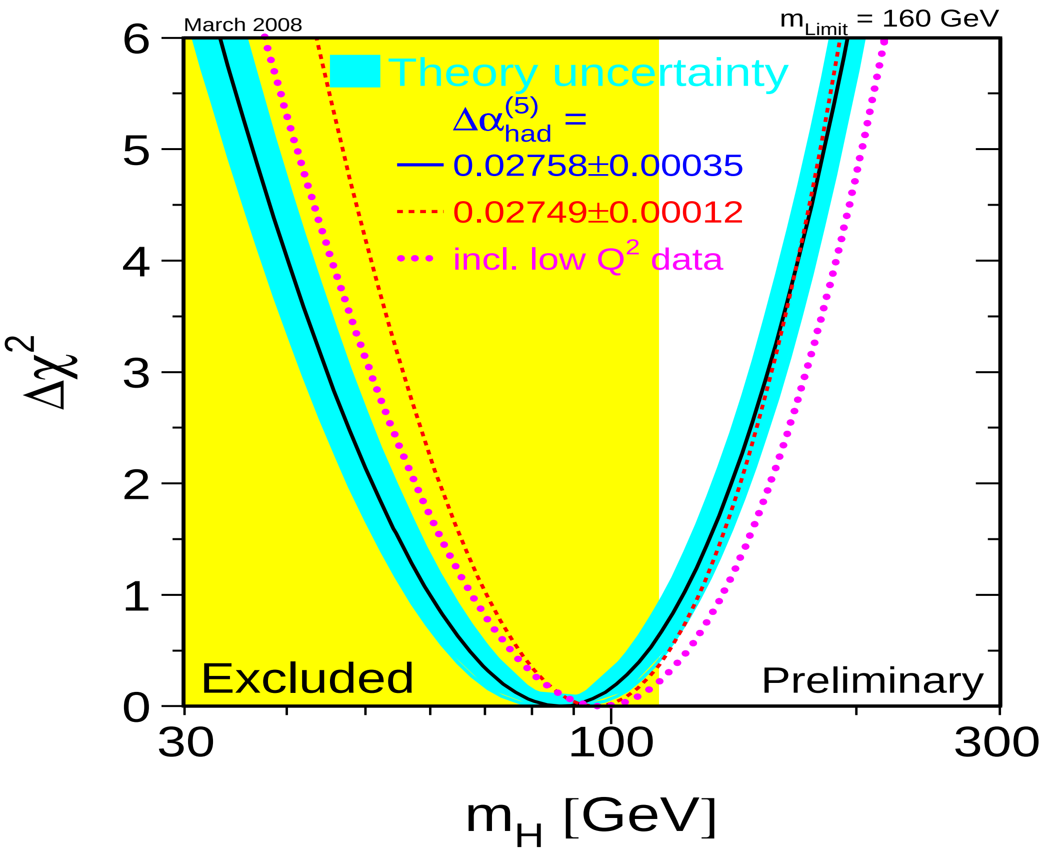
ALL LEP AND SLC SM OBSERVABLES, EXCEPT A_{FB}^b, POINT TO LIGHT HIGGS-BOSON MASS

• Higgs search is a central target of the experiments at Tevatron; current sensitivity to various production & decay channels restricted by luminosity

• Discovering Higgs & determining its properties is one of the principal reasons for building LHC

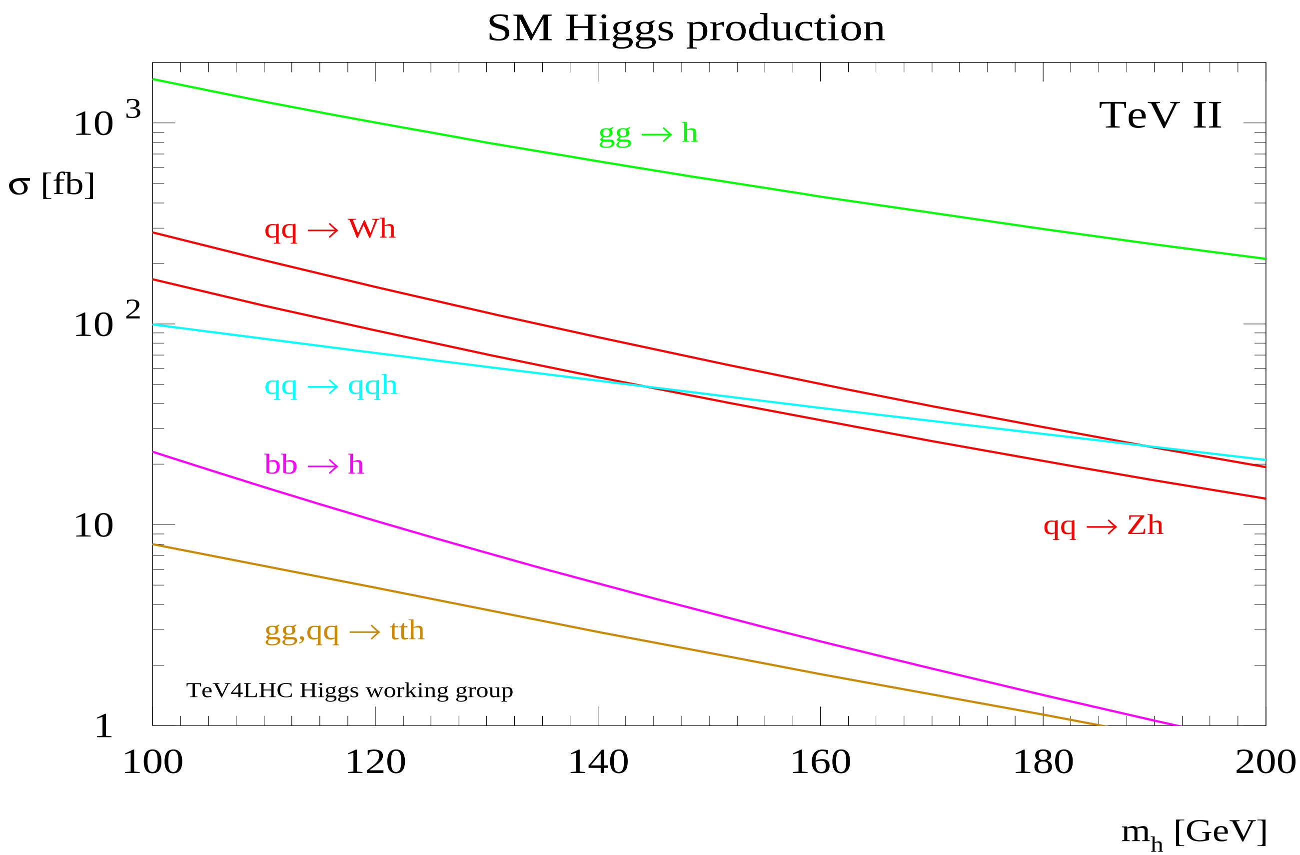
Direct and Indirect bounds on the Higgs Boson Mass

[LEP Electroweak Working Group: http://lepewwg.web.cern.ch/LEPEWWG/)]



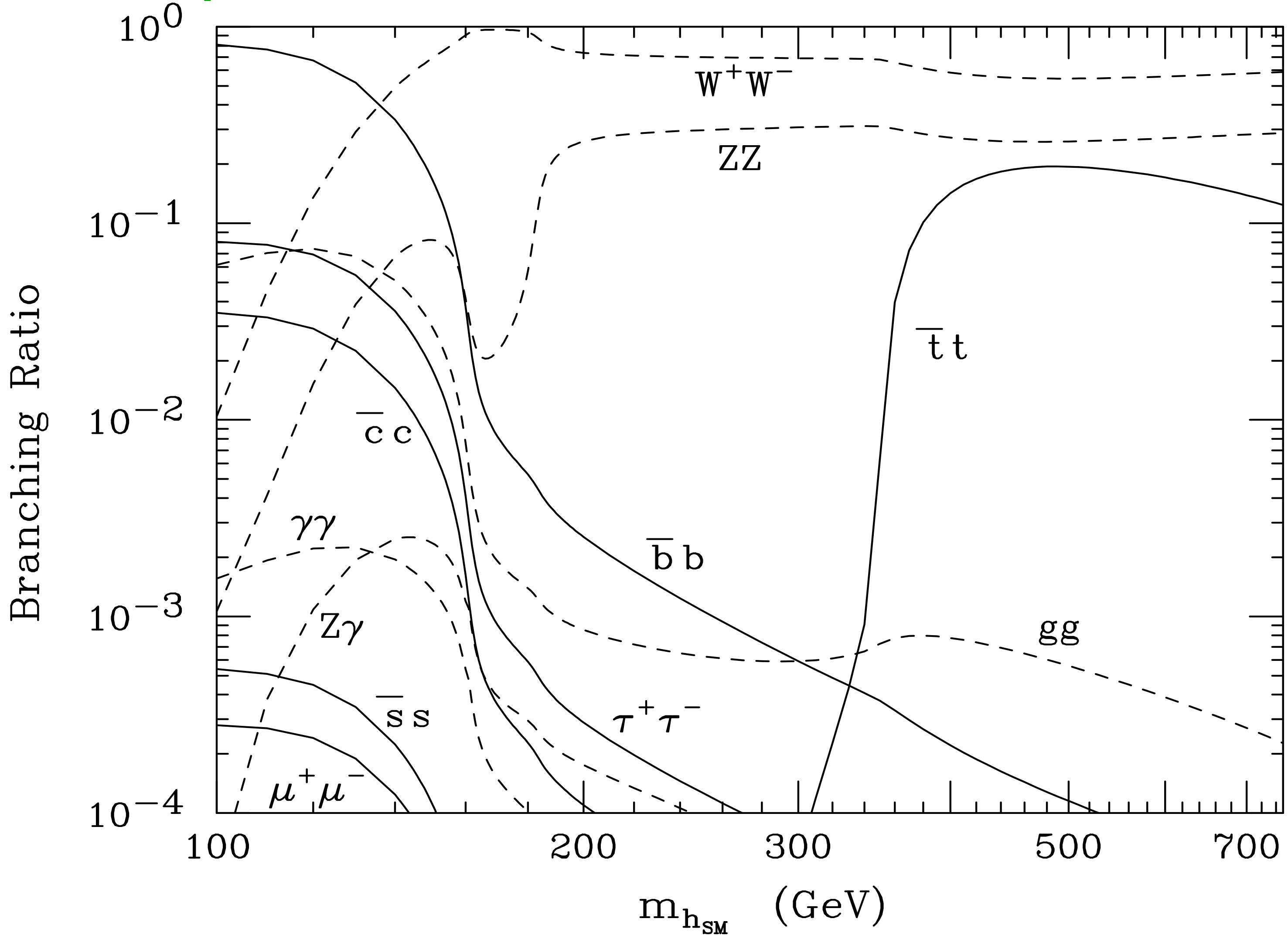
SM Higgs X-section at the Tevatron

[T. Hahn et al., hep-ph/0607308 & http://maltoni.home.cern.ch/maltoni/TeV4LHC/]

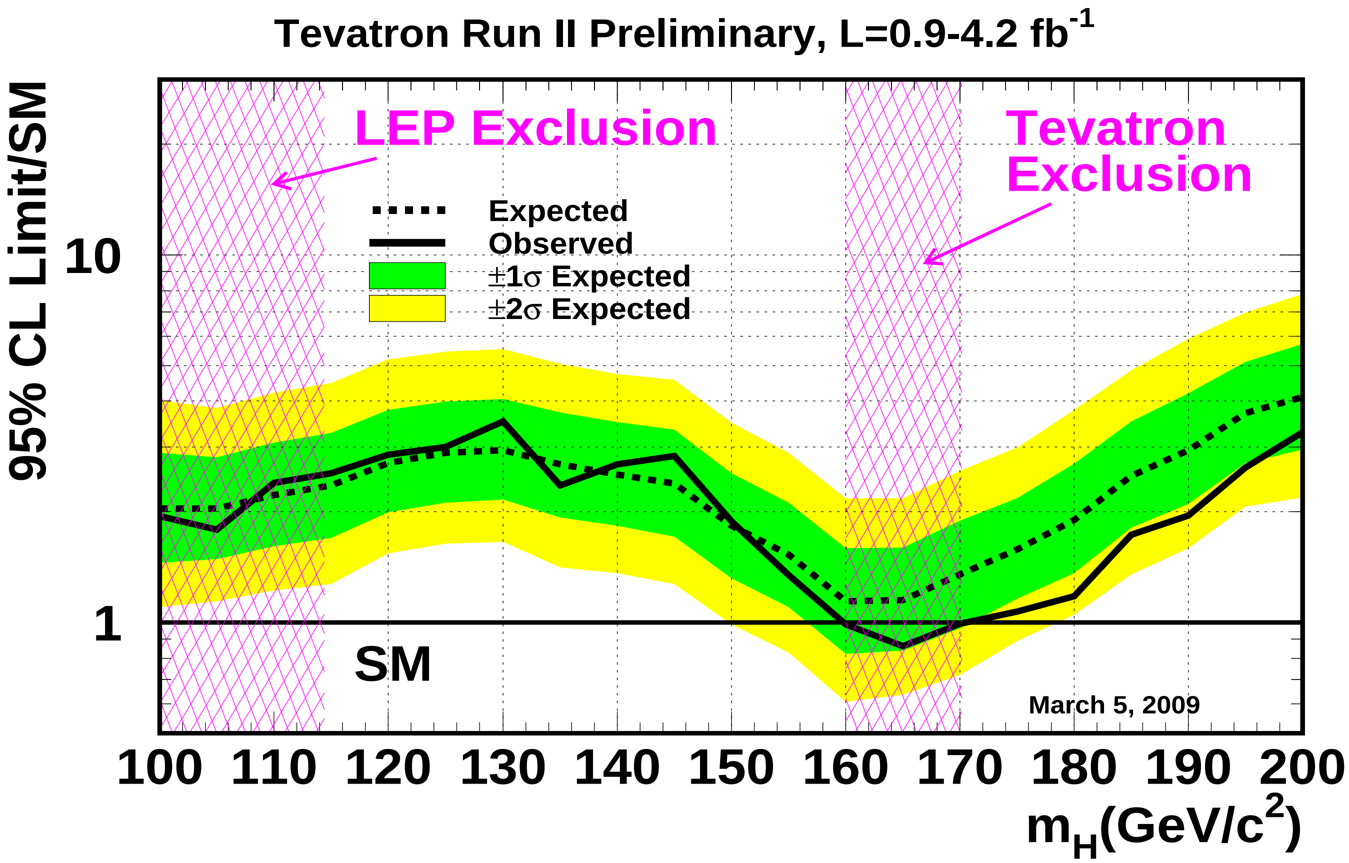


SM Higgs decay branching ratios

[Carena & Haber]

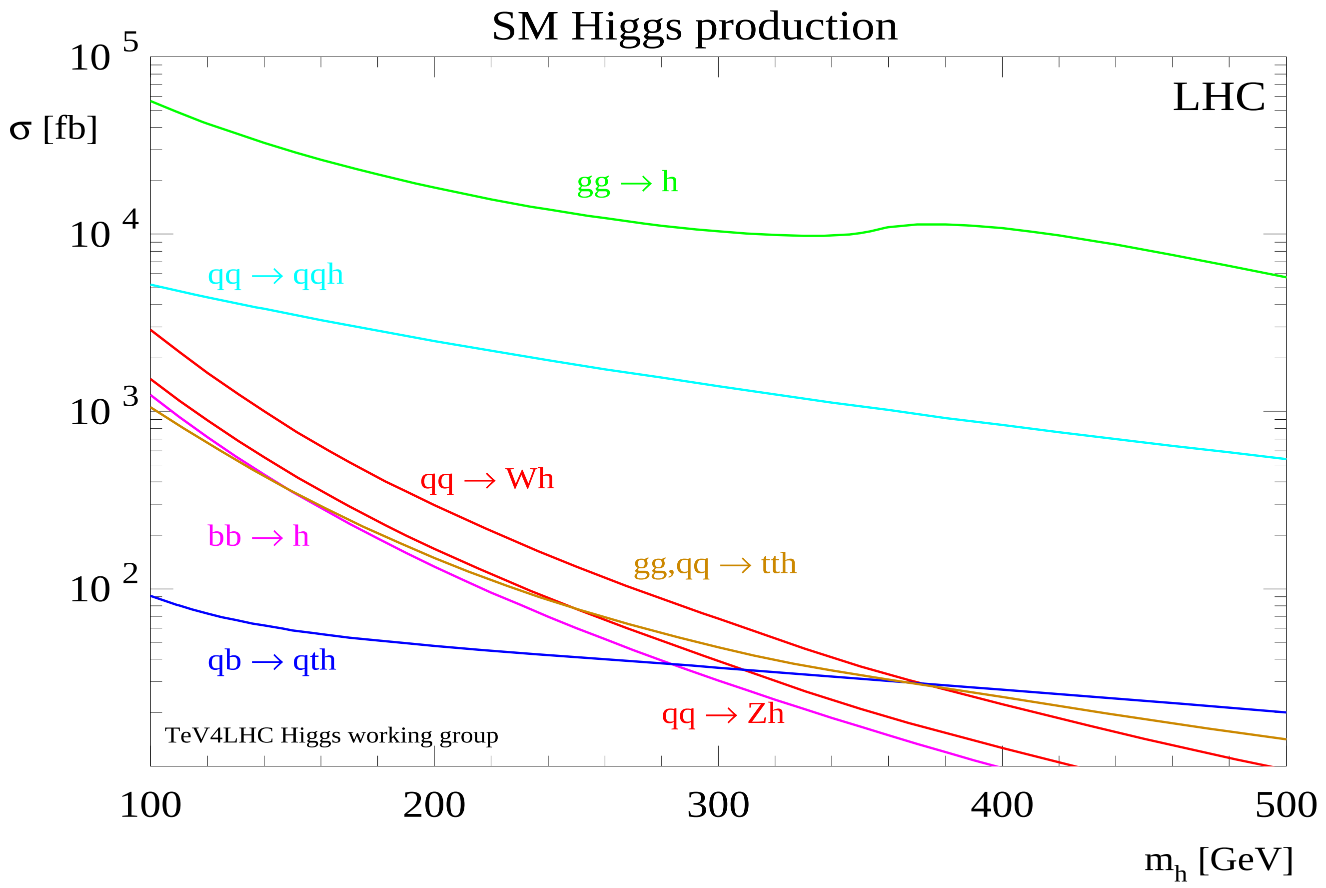


CDF and D0 limits on the SM Higgs Boson



SM Higgs X-section at the LHC

[T. Hahn et al., hep-ph/0607308 & http://maltoni.home.cern.ch/maltoni/TeV4LHC/]



Gauge couplings in the SM

- The gauge couplings g_i ($i = 1, 2, 3$ for $U(1)_Y,\ SU(2)_I$ and $SU(3)_c$) are energy-dependent due to the renormalization effects

$$\begin{aligned}
\mu\frac{d}{d\mu}g_3(\mu) &= -\frac{g_3^2}{16\pi^2}(11-\frac{2}{3}n_f)+O(g_3^5)\\
\mu\frac{d}{d\mu}g_2(\mu) &= -\frac{g_2^2}{24\pi^2}(11-n_f)+O(g_2^5)\\
\mu\frac{d}{d\mu}g_1(\mu) &= -\frac{g_1^2}{4\pi^2}(-\frac{10}{9}n_f)+O(g_1^5)
\end{aligned}$$

- To lowest order in the respective couplings

$$\alpha_i(\mu)^{-1} = \alpha_i(M)^{-1} + \frac{1}{2\pi}b_i^{\rm SM}\ln(\frac{M}{\mu})$$

For $n_f = 6$, $b_i^{\rm SM} = (b_1^{\rm SM}, b_2^{\rm SM}, b_3^{\rm SM}) = (41/10, -19/6, -7)$; M is some reference scale

- Thanks to precise electroweak and QCD measurements at LEP, Tevatron and HERA, values of the three coupling constants well measured at the scale $Q = M_Z$

$$\begin{aligned}
\alpha_s(M_Z) &= \alpha_3(M_Z) = 0.1187 \pm 0.002 \quad \Longrightarrow \quad \Lambda_{\rm QCD}^{(5)} = 217^{+25}_{-23}\ {\rm MeV}\\
\alpha_{\rm em}^{-1}(M_Z) &= \frac{5}{3}\alpha_1^{-1}(M_Z) + \alpha_2^{-1}(M_Z) = 128.91 \pm 0.02\\
\alpha_2(M_Z) &= \frac{\alpha_{\rm em}}{\sin^2(\theta)}(M_Z) \ \ {\rm with} \ \ \sin^2(\theta)(M_Z) = 0.2318 \pm 0.0005
\end{aligned}$$

Gauge coupling unification in supersymmetric theories

- Unification of the gauge couplings (of the strong, electromagnetic and weak interactions) in grand unified theories widely anticipated [Dimopoulos, Raby, Wilczek; Langacker et al.; Ellis et al.; Amaldi et al.;...]
- However, unification does not work in $SU(5)$; works in supersymmetric $SU(5)$ but also in $SO(10)$, as both have an additional scale: $\Lambda_{\rm S}$ for SUSY phenomenologically of $O(1)$ TeV; grand unification works in $SO(10)$ due to an additional scale $M(W_R)$ for the right-handed weak bosons; both cases yielding the Grand Unification Scale $\Lambda_G = O(3 \times 10^{16})$ GeV
- In SUSY, the running of $\alpha_i(\mu)$ changes according to

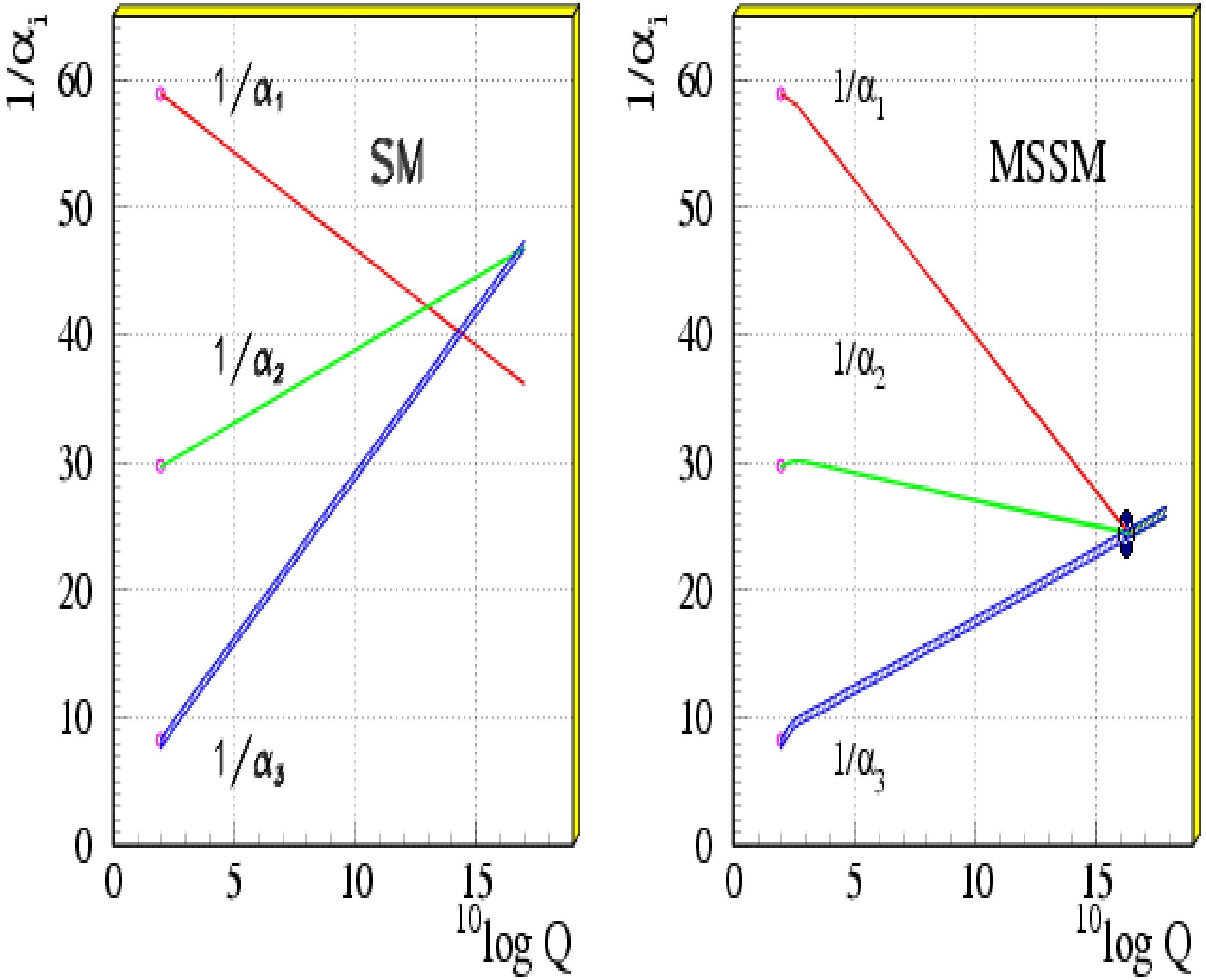
$$\begin{aligned}\alpha_i(\mu)^{-1} &= \left(\frac{1}{\alpha_i^0(\Lambda_{\rm G})} + \frac{1}{2\pi} b_i^{\rm S} \ln(\frac{\Lambda_{\rm G}}{\mu})\right)\theta(\mu - \Lambda_{\rm S}) \\ &+ \left(\frac{1}{\alpha_i^0(\Lambda_S)} + \frac{1}{2\pi} b_i^{\rm SM} \ln(\frac{\Lambda_S}{\mu})\right)\theta(\Lambda_{\rm S} - \mu)\end{aligned}$$

  with $b_i^{\rm SM} = (b_1^{\rm SM}, b_2^{\rm SM}, b_3^{\rm SM}) = (41/10, -19/6, -7)$ below the supersymmetric scale, and $b_i^{\rm S} = (b_1^{\rm S}, b_2^{\rm S}, b_3^{\rm S}) = (33/5, 1, -3)$ above the supersymmetric scale
- The unified coupling α_G is related to $\alpha_i(M_Z)$ by

$$\alpha_i^{-1}(M_Z) = \alpha_G^{-1} + b_i t_G$$

  with $t_G = \frac{1}{2\pi} \ln \frac{M_G}{M_Z} \sim 5.32$ and $\alpha_G^{-1} = 23.3$

Gauge coupling unification in the SM and MSSM



Summary

- Nature does not read papers on free field theories. Learn Quantum Field Theory!
- Applications of QCD have permeated in practically all the branches of particle and nuclear physics and there is a lot of scope to make solid contributions in this area
- The current experimental and theoretical thrust is on understanding the mechanism of electroweak symmetry breaking (Higgs mechanism or alternatives)
- There exist strong hints for Supersymmetry (gauge coupling unification, candidates for dark matter); Higgs mass stability also requires Supersymmetry
- LHC is the next high energy frontier. Let us see what it brings to us!
- All those entering the field now, my message is:

You are faced with insurmountable oppurtunities! Go, get them!!

Backup Slides

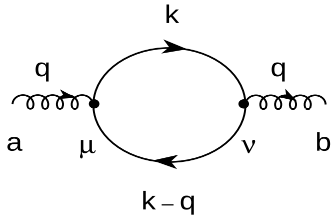
- Dimensional regularization
- Renormalization: QED

Regularization of Loop Integrals

General Remarks:

- In the lowest order (Born Approximation), straightforward to calculate scattering amplitudes, given the Feynman rules. However, a reliable theory must be stable against Quantum Corrections (Loops)
- Loop integrals are often divergent; as QCD is a gauge theory, need to have a gauge-invariant method of regularizing these divergences; the most popular method is Dimensional Regularization
- Conceptually, the dependence on the regulator is eliminated by absorbing it into a redefinition of the coupling constant, of the quark mass(es), and the wave function renormalization of the quark and gluon fields $(Z_1, Z_2, Z_3, \delta m = m - m_0)$
- If we neglect the quark masses (a good approximation for the light quarks u and d), then QCD has a coupling constant (g_s) but no scale. Quantum effects generate a scale, Λ_{QCD}
- QCD by itself does not determine Λ_{QCD}, much the same way as it does not determine the quark masses. These fundamental parameters have to be determined by theoretical consistency and experiments

Dimensional Regularization



- Let us consider the self-energy gluon loop above

$$i\,\Pi^{\mu\nu}_{ab}(q) \;=\; -g_s^2\delta_{ab}T_F\int\frac{d^4k}{(2\pi)^4}\frac{\mathrm{Tr}[\gamma^\mu \not{k}\gamma^\nu(\not{k}-\not{q})]}{k^2(k-q)^2}$$

- The problem appears in the momentum integration, which is divergent
  $\sim\int d^4k(1/k^2)\longrightarrow\infty$

- Regularize the loop integrals through *dimensional regularization*; the calculation is performed in $D=4-2\epsilon$ dimensions. For $\epsilon\neq 0$ the resulting integral is well defined:

$$\int\frac{d^Dk}{(2\pi)^D}\frac{k^\alpha(k-q)^\beta}{k^2(k-q)^2} = \frac{-i}{6(4\pi)^2}\left(\frac{-q^2}{4\pi}\right)^\epsilon\Gamma(-\epsilon)\left(1-\frac{5}{3}\epsilon\right)\left\{\frac{q^2g^{\alpha\beta}}{2(1+\epsilon)}+q^\alpha q^\beta\right\}$$

- The ultraviolet divergence of the loop appears at $\epsilon=0$, through the pole of the Gamma function

$$\Gamma(-\epsilon) \;=\; -\frac{1}{\epsilon}-\gamma_E+\mathcal{O}(\epsilon^2)\,, \qquad\qquad \gamma_E=0.577215\ldots$$

Dimensional Regularization Contd.

- Introduce an arbitrary energy scale μ and write

$$\mu^{2\epsilon}\left(\frac{-q^2}{4\pi\mu^2}\right)^{\epsilon}\Gamma(-\epsilon) = -\mu^{2\epsilon}\left\{\frac{1}{\epsilon} + \gamma_E - \ln 4\pi + \ln\left(-q^2/\mu^2\right) + \mathcal{O}(\epsilon)\right\}$$

- Written in this form, one has a dimensionless quantity $(-q^2/\mu^2)$ inside the logarithm. The contribution of the loop diagram can be written as

$$\Pi^{\mu\nu}_{ab} = \delta_{ab}\left(-q^2 g^{\mu\nu} + q^\mu q^\nu\right)\Pi(q^2)$$

which is ultraviolet divergent, with

$$\Pi(q^2) = -\frac{4}{3}T_F\left(\frac{g_s\mu^\epsilon}{4\pi}\right)^2\left\{\frac{1}{\epsilon} + \gamma_E - \ln 4\pi + \ln\left(-q^2/\mu^2\right) - \frac{5}{3} + \mathcal{O}(\epsilon)\right\}.$$

- While divergent, and hence the self-energy contribution remains undetermined, we know now how the effect changes with the energy scale

- Fixing $\Pi(q^2)$ at some reference momentum transfer q_0^2, the result is known at any other scale:

$$\Pi(q^2) = \Pi(q_0^2) - \frac{4}{3}T_F\left(\frac{g_s}{4\pi}\right)^2\ln\left(q^2/q_0^2\right)$$

- Split the self-energy contribution into a divergent piece and a finite q^2-dependent term

$$\boldsymbol{\Pi(q^2) \equiv \Delta\Pi_\epsilon(\mu^2) + \Pi_R(q^2/\mu^2)}$$

- This separation is ambiguous, as the splitting can be done in many different ways. A given choice defines a scheme:

$$\Delta\Pi_\epsilon(\mu^2) = \begin{cases} -\frac{T_F}{3\pi}\frac{g_s^2}{4\pi}\mu^{2\epsilon}\left[\frac{1}{\epsilon}+\gamma_E-\ln(4\pi)-\frac{5}{3}\right] & (\mu \text{ scheme}) \\ -\frac{T_F}{3\pi}\frac{g_s^2}{4\pi}\mu^{2\epsilon}\frac{1}{\epsilon} & (\text{MS scheme}) \\ -\frac{T_F}{3\pi}\frac{g_s^2}{4\pi}\mu^{2\epsilon}\left[\frac{1}{\epsilon}+\gamma_E-\ln(4\pi)\right] & (\overline{\text{MS}} \text{ scheme}) \end{cases}$$

$$\Pi_R(q^2/\mu^2) = \begin{cases} -\frac{T_F}{3\pi}\frac{g_s^2}{4\pi}\ln\left(-q^2/\mu^2\right) & (\mu \text{ scheme}) \\ -\frac{T_F}{3\pi}\frac{g_s^2}{4\pi}\left[\ln\left(-q^2/\mu^2\right)+\gamma_E-\ln(4\pi)-\frac{5}{3}\right] & (\text{MS scheme}) \\ -\frac{T_F}{3\pi}\frac{g_s^2}{4\pi}\left[\ln\left(-q^2/\mu^2\right)-\frac{5}{3}\right] & (\overline{\text{MS}} \text{ scheme}) \end{cases}$$

- In the μ scheme, $\Pi(-\mu^2)$ is used to define the divergent part

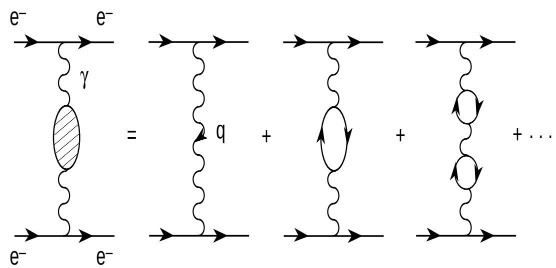
- In the MS (minimal subtraction) scheme, one subtracts only the divergent $1/\epsilon$ term

- In the $\overline{\text{MS}}$ (modified minimal subtraction schemes), one puts also the $\gamma_E - \ln(4\pi)$ factor into the divergent part

- Note, the logarithmic q^2 dependence in $\Pi_R(q^2/\mu^2)$ is always the same in these schemes

Renormalization: QED

- Recall: A QFT is called renormalizable if all ultraviolet divergences can be absorbed through a redefinition of the original fields, masses and couplings
- Let us consider the electromagnetic interaction between two electrons



- At one loop, the QED photon self-energy contribution is:

$$\Pi(q^2) = -\frac{4}{3}\left(\frac{e\mu^\epsilon}{4\pi}\right)^2 \left\{\frac{1}{\epsilon} + \gamma_E - \ln 4\pi + \ln\left(-q^2/\mu^2\right) - \frac{5}{3} + \mathcal{O}(\epsilon)\right\}$$

- The scattering amplitude takes the form

$$T(q^2) \sim -J^\mu J_\mu \frac{e^2}{q^2}\left\{1 - \Pi(q^2) + \ldots\right\}$$

- J^μ denotes the electromagnetic fermion current. At lowest order, $T(q^2) \sim \alpha/q^2$ with $\alpha = e^2/(4\pi)$

- The divergent correction from the quantum loops can be reabsorbed into a redefinition of the electromagnetic coupling:

$$\frac{\alpha_0}{q^2}\left\{1-\Delta\Pi_\epsilon(\mu^2)-\Pi_R(q^2/\mu^2)\right\} \equiv \frac{\alpha_R(\mu^2)}{q^2}\left\{1-\Pi_R(q^2/\mu^2)\right\}$$

$$\alpha_R(\mu^2) = \alpha_0\,(1-\Delta\Pi_\epsilon(\mu^2)) = \alpha_0\left\{1+\frac{\alpha_0}{3\pi}\mu^{2\epsilon}\left[\frac{1}{\epsilon}+C_{\text{scheme}}\right]+\ldots\right\}$$

where $\alpha_0 \equiv \frac{e_0^2}{4\pi}$, and e_0 is the *bare* coupling appearing in the QED Lagrangian and is not an observable

- The scattering amplitude written in terms of α_R is finite and the resulting cross-section can be compared with experiment; thus, one actually measures the renormalized coupling α_R

- Both $\alpha_R(\mu^2)$ and the renormalized self-energy correction $\Pi_R(q^2/\mu^2)$ depend on μ, but the physical scattering amplitude $T(q^2)$ is μ-independent: $(Q^2 \equiv -q^2)$

$$T(q^2) = 4\pi\,J^\mu J_\mu\,\frac{\alpha_R(Q^2)}{Q^2}\left\{1+\frac{\alpha_R(Q^2)}{3\pi}C'_{\text{scheme}}+\cdots\right\}$$

- The quantity $\alpha(Q^2) \equiv \alpha_R(Q^2)$ is called the QED *running coupling*

- The usual fine structure constant $\alpha = 1/137$ is defined through the classical Thomson formula and corresponds to a very low scale $Q^2 = -m_e^2$

- The scale dependence of $\alpha(Q^2)$ is regulated by the so-called β function

$$\mu \frac{d\alpha}{d\mu} \equiv \alpha \, \beta(\alpha) \, ; \qquad \beta(\alpha) = \beta_1 \frac{\alpha}{\pi} + \beta_2 \left(\frac{\alpha}{\pi}\right)^2 + \cdots$$

- At the one-loop level, β reduces to its first coefficient

$$\beta_1^{\mathrm{QED}} = \frac{2}{3}$$

- The solution of the differential equation for α is:

$$\alpha(Q^2) = \frac{\alpha(Q_0^2)}{1 - \frac{\beta_1 \alpha(Q_0^2)}{2\pi} \ln\left(Q^2/Q_0^2\right)}$$

- Since $\beta_1 > 0$, the QED running coupling increases with the energy scale: $\alpha(Q^2) > \alpha(Q_0^2)$ if $Q^2 > Q_0^2$

- The value of α relevant for high Q^2, measured at LEP, confirms this: $\alpha(M_Z^2)_{\overline{\mathrm{MS}}} \simeq 1/128$