Integrability and the Zero Curvature Representation of the Sinh-Gordon Equation

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DESY Summer Students Programme

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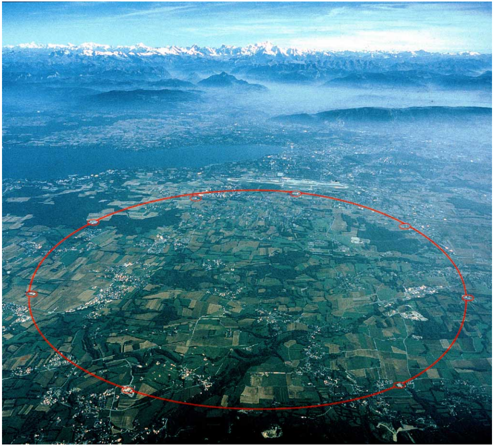
Physics is about experiments...



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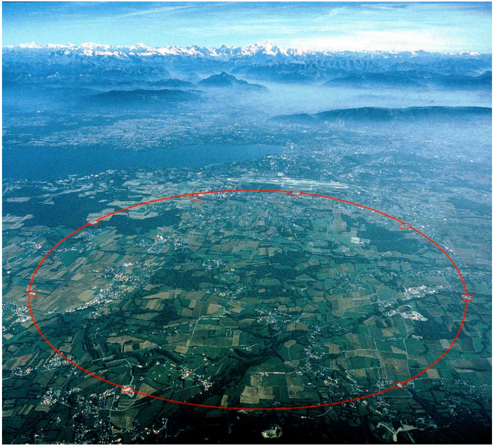

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Outline

Aim of this talk: a short introduction to integrability and why the zero curvature representation is important, and the construction of an infinite number of conserved quantities for the sinh-Gordon equation.

Contents:

Examples of integrable systems

- Euler's top, Coulomb potential, harmonic oscillator, KdV equation, Heisenberg spin chain, Nahm equations, Landau-Lifshitz equation, (anti-)self-dual Yang Mills equations, nonlinear Schrodinger equation, sine-Gordon and sinh-Gordon equations.
- Could also be called "exactly solvable models"
- Physical motivation for studying: occur in different areas of physics, eg classical mechanics, magnetism, gauge theories. Admit exact solutions, sometimes solitons (particle-like).

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- For a system with n degrees of freedom we need n conserved quantities which are compatible in the sense that their Poisson brackets commute (Hamiltonian formalism)
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Sinh-Gordon equation

- One scalar field $\phi = \phi(x, t)$
- Defined on a two-dimensional space-time with periodic boundary conditions: $\phi(x, t) = \phi(x + R, t)$
- The sinh-Gordon Hamiltonian is

$$H = \int_0^R dx \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\phi')^2 + m^2 \cosh 4\phi \right)$$

where $\pi = \dot{\phi}$.
- The equation of motion is

$$\ddot{\phi} - \phi'' = -4m^2 \sinh 4\phi$$

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Sinh-Gordon rewritten

- Key observation: the sinh-Gordon equation $\ddot{\phi} - \phi'' = -4m^2 \sinh 4\phi$ is equivalent to

$$[\partial_t - V, \partial_x - U] = 0 \Leftrightarrow V' - \dot{U} + VU - UV = 0$$

for the two two-by-two matrices

$$U(x,t,\lambda) = \begin{pmatrix} \dot{\phi} & m\lambda^{-1}e^{2\phi} + m\lambda e^{-2\phi} \\ m\lambda^{-1}e^{-2\phi} + m\lambda e^{2\phi} & -\dot{\phi} \end{pmatrix}$$

$$V(x,t,\lambda) = \begin{pmatrix} \phi' & -m\lambda^{-1}e^{2\phi} + m\lambda e^{-2\phi} \\ -m\lambda^{-1}e^{-2\phi} + m\lambda e^{2\phi} & -\phi' \end{pmatrix}$$

What are these matrices?

- We interpret U and V as auxiliary gauge fields. At each point (x, t) we imagine U and V acting on a two-dimensional vector space.
- Can use U and V to define a generalised (covariant) differentiation of vectors, $D_x\psi = (\partial_x - U)\psi$, $D_t\psi = (\partial_t - V)\psi$.
- This also gives us a method for **parallel transporting** vectors from (x, t) to (x', t'); the parallel transport of a vector ψ along a curve γ is defined by the conditions that

$$(\partial_x - U)\psi = 0 \qquad (\partial_t - V)\psi = 0$$

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Parallel transport operators

- Consider just the x-direction, path from y to x, seek a matrix $T(x,y)$ such that for a vector ψ, $\psi' = T\psi$ satisfies $(\partial_x - U)\psi' = 0$, i.e.

$$(\partial_x - U)T = 0 \qquad T(x,x) = I$$

- The solution is

$$T(x,y,\lambda) = \mathcal{P}\exp\left(\int_y^x dx\, U(x,\lambda)\right)$$

- The **monodromy matrix** is the operator of parallel transport for a path starting at $x=0$ and finishing at $x=R$.

$$M(\lambda) = \mathcal{P}\exp\left(\int_0^R dx\, U(x,\lambda)\right)$$

- Similarly in t-direction, have $S(t_1,t_2,\lambda) = \mathcal{P}\int_{t_1}^{t_2} dt\, V(t,\lambda)$

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What is curvature?

- The **curvature** of the connection is the commutator $[\partial_t - V, \partial_x - U]$. The **zero curvature condition** is the condition that $[\partial_t - V, \partial_x - U] = 0$.
- If ψ_1 and ψ_2 are the vectors obtained by parallel transporting some vector ψ along two different paths with the same endpoints, then the curvature measures the difference between ψ_1 and ψ_2.
- A simplified way to see this is to consider the following path:

$$\psi \rightarrow \psi + (\partial_x - U)\psi dx$$
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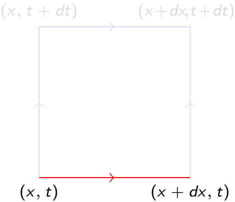

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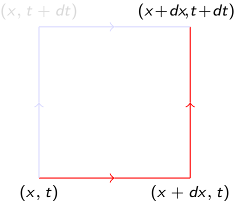

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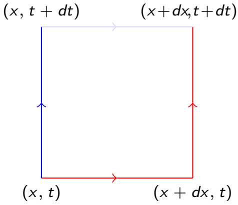

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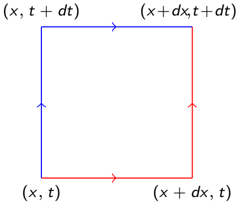

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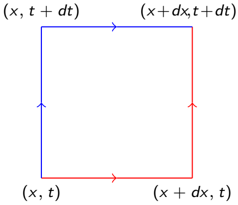

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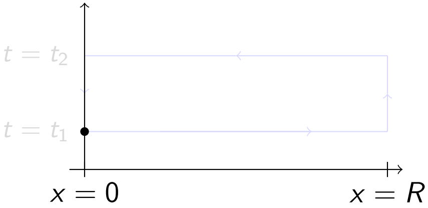
Monodromy matrix contd

In particular, parallel transport around a closed path is the identity.

For the path shown we have:



$$M(\lambda)|_{t_1} S(R) M^{-1}(\lambda)|_{t_2} S^{-1}(R) = I$$

$$\Rightarrow M(\lambda)|_{t_1} = S(R) M(\lambda)|_{t_2} S^{-1}(R)$$

$$\Rightarrow M(\lambda)|_{t_1} \sim M(\lambda)|_{t_2} \Rightarrow \operatorname{tr} M(\lambda)|_{t_1} = \operatorname{tr} M(\lambda)|_{t_2}$$

So the trace of the monodromy matrix is constant in time, and can be used to generate conserved quantities!

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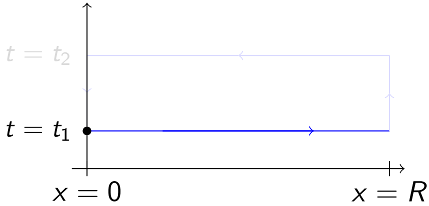

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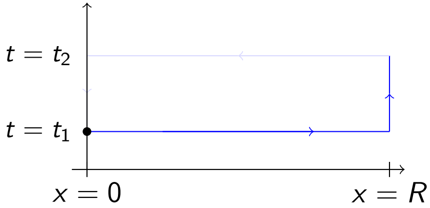

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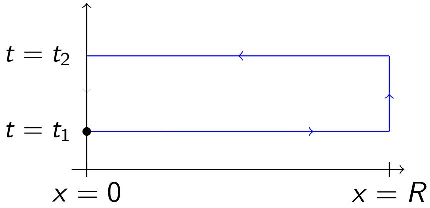

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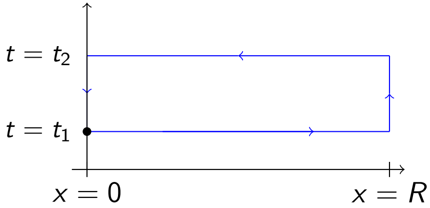

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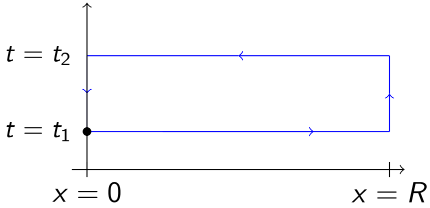

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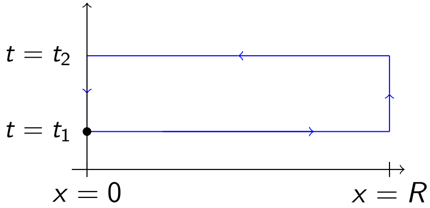

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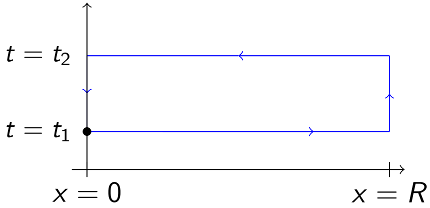

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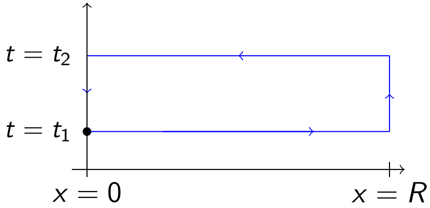

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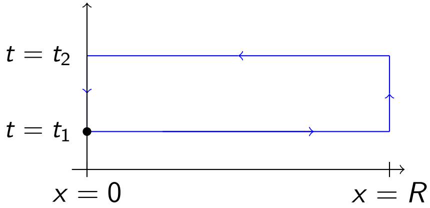

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Constructing the conserved quantities

- The idea is to **gauge transform** U into a simple form.
- Gauge transformations are transformations of the gauge fields U, V which leave the physical situation (sinh-Gordon equation) unchanged.
- We gauge transform via invertible matrices g acting by conjugation:

$$\partial_x - U \to g^{-1}(\partial_x - U)g = \partial_x + g^{-1}\partial_x g - g^{-1}Ug$$

- We can find gauge transformations so that U becomes diagonal, then it is straightforward to calculate $M(\lambda) = \mathcal{P}\exp\left(\int_0^R dx\, U(x,\lambda)\right)$. We then look at the trace for $\lambda \to \infty, \lambda \to 0$, find expansions

$$\log \operatorname{tr} \mathcal{M}(\lambda) \to \lambda^{\pm 1} R - \sum_{k=1}^{\infty} \lambda^{\mp k} H_k^{\pm}$$

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Conserved quantities - results

- For instance we find

$$H_1^{\pm} = -\int_0^R dx \left(\frac{1}{2}(\dot{\phi} \pm \phi')^2 + m^2 \cosh 4\phi \right)$$

- Note that the Hamiltonian is given by

$$H = -\frac{1}{2}(H_1^+ + H_1^-)$$

and the momentum $P = \int_0^R dx\, \dot{\phi}\phi'$ by

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Conserved quantities - what else can we do?

- Similar methods apply to a broad class of systems which admit a zero curvature representation.
- Sinh-Gordon equation is the simplest version of an affine Toda field theory - an infinite family of field equations with Hamiltonian

$$\mathcal{H} = \int_0^R dx \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\phi')^2 + 2m^2 \sum_{i=0}^{r} \frac{1}{\alpha_i^2} e^{2\alpha_i \cdot \phi} \right)$$

  where $\phi = (\phi_i)$ is an r-component vector of r scalar fields, π consists of the conjugate momenta, and the α_i are the simple roots of an affine Lie algebra (special vectors characterising the algebra uniquely)
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Conclusion

- Integrability gives us a powerful tool for analysing many systems in both mathematics and physics
- Characteristics of integrability: conserved quantities, zero curvature representation, Hamiltonian structure...

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